② TRIGONOMETRY:

MATH-I/1

(A) INTRODUCTION:

9 Trigonometry is the branch of mathematics which deals with the study of the relationship between the sides
10 and angles of a triangle. There are three system of measurements of angle i.e

11 (i) sexagesimal system:

1 right angle $= 90^\circ$

12 $1^\circ = 60$ sexagesimal minutes $(60')$

~~1 minute (1') = 60 sexagesimal minutes~~ 60

1 $1' = 60$ sexagesimal second $(60'')$

This system is called English system.

2 (ii) centisimal system:

3 1 right angle $= 100$ grade (100^g)

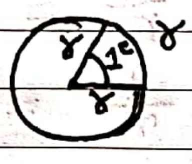
$1^g = 100'$

4 $1' = 100''$

Note that 1 right angle $= 90^\circ = 100^g$.

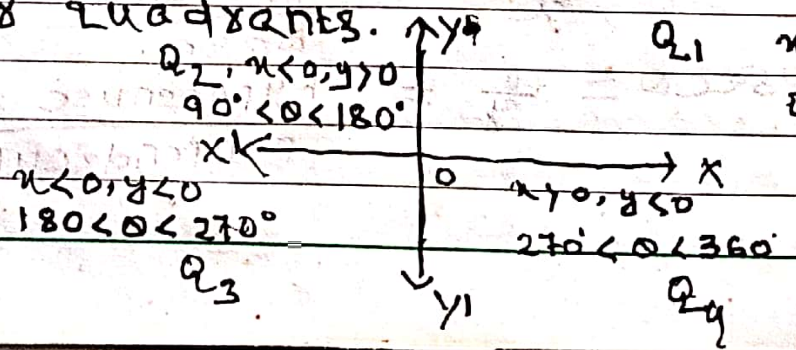
5 (III) circular system:

(a) A radian is the angle subtended at the centre
6 of a circle by an arc whose length is equal to the radius of that circle. It is denoted by
7 1^c.



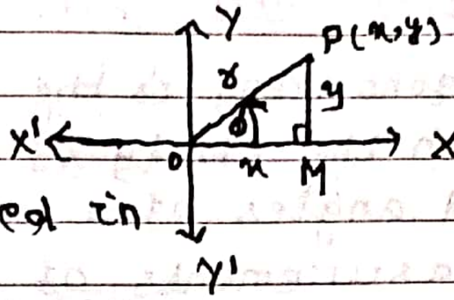
(b) $\boxed{\pi^c = 180^\circ = 200^g}$

(B) QUADRANTS: X-axis & y-axis divide the plane into four quadrants.



C: TRIGONOMETRICAL RATIOS:

consider a line OP making an angle θ with +ve direction of x-axis measured in anticlockwise direction.



Let $\overline{PM} \perp$ x-axis. Then $OM = x$, $PM = y$,

Let $OP = r = \sqrt{x^2+y^2}$

OPM is a right angled triangle.

PM = perpendicular, OM = base, OP = hypotenuse.

using these 3 sides we have six ratios known as Trigonometrical ratios. ~~where~~ These are defined in point (ii).

(i)

Full name	symbol
sine	sin.
co-sine	cos.
tangent	tan.
co-tangent	cot.
secant	sec.
cosecant	cosec.

(ii) $\sin\theta = \frac{y}{r} = \frac{\text{perpendicular}}{\text{hypotenuse}}$.

$\cos\theta = \frac{x}{r} = \frac{\text{base}}{\text{hypotenuse}}$

$\tan\theta = \frac{y}{x} = \frac{\text{perpendicular}}{\text{base}}$.

$\cot\theta = \frac{x}{y} = \frac{\text{base}}{\text{perpendicular}}$

$\sec\theta = \frac{r}{x} = \frac{\text{hypotenuse}}{\text{base}}$.

$\mathrm{cosec}\,\theta = \frac{r}{y} = \frac{\text{hypotenuse}}{\text{perpendicular}}$.

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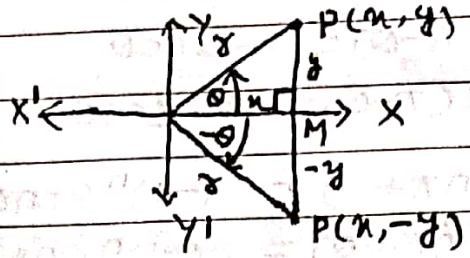
D: TRIGONOMETRIC RATIO OF ANGLE $(-\theta)$

When the angle θ is measured in clock-wise direction with +ve direction of x-axis, it is represented by $-\theta$. Here $(-)$ sign indicates the direction of measurement of angle.



Note that

(a) $\sin(-\theta) = -\sin\theta$

(b) $\cos(-\theta) = \cos\theta$

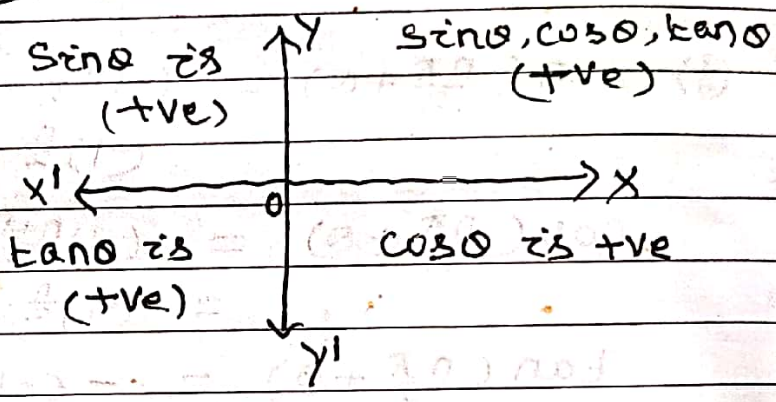
(c) $\tan(-\theta) = -\tan\theta$

(d) $\cot(-\theta) = -\cot\theta$

(e) $\sec(-\theta) = \sec\theta$

(f) $\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$

E: ASTC RULE:



A = All (+ve)

S = $\sin\theta$ (+ve)

T = $\tan\theta$ (+ve)

C = $\cos\theta$ (+ve)

Out of three t-ratios $\sin\theta$, $\cos\theta$, $\tan\theta$

In Q_1, all are (+ve)

In Q_2, $\sin\theta$ is +ve

In Q_3, $\tan\theta$ is +ve

In Q_4, $\cos\theta$ is +ve

Consequently sign of these three T-ratios gives sign of other three T-ratios.

(F) T-Ratios of Angle $\frac{\pi}{2}-\theta$, $\frac{\pi}{2}+\theta$, $\pi-\theta$, $\pi+\theta$, $n\pi+\theta$, $n\pi-\theta$, $\frac{n\pi}{2}+\theta$:

(a) $\sin(\frac{\pi}{2}-\theta) = \cos\theta$

$\cos(\frac{\pi}{2}-\theta) = \sin\theta$

$\tan(\frac{\pi}{2}-\theta) = \cot\theta$

$\cot(\frac{\pi}{2}-\theta) = \tan\theta$

$\sec(\frac{\pi}{2}-\theta) = \operatorname{cosec}\theta$

$\operatorname{cosec}(\frac{\pi}{2}-\theta) = \sec\theta$

(b) $\sin(\frac{\pi}{2}+\theta) = \cos\theta$

$\cos(\frac{\pi}{2}+\theta) = -\sin\theta$

$\tan(\frac{\pi}{2}+\theta) = -\cot\theta$

$\cot(\frac{\pi}{2}+\theta) = -\tan\theta$

$\sec(\frac{\pi}{2}+\theta) = -\operatorname{cosec}\theta$

$\operatorname{cosec}(\frac{\pi}{2}+\theta) = \operatorname{cosec}\theta$

(c) $\sin(\pi-\theta) = \sin\theta$
$\cos(\pi-\theta) = -\cos\theta$
$\tan(\pi-\theta) = -\tan\theta$
$\cot(\pi-\theta) = -\cot\theta$
$\sec(\pi-\theta) = -\sec\theta$
$\operatorname{cosec}(\pi-\theta) = \operatorname{cosec}\theta$

(d) $\sin(\pi+\theta) = -\sin\theta$
$\cos(\pi+\theta) = -\cos\theta$
$\tan(\pi+\theta) = \tan\theta$
$\cot(\pi+\theta) = \cot\theta$
$\sec(\pi+\theta) = -\sec\theta$
$\operatorname{cosec}(\pi+\theta) = -\operatorname{cosec}\theta$

(e) $\sin(n\pi+\theta) = (-1)^n \sin\theta$
$\cos(n\pi+\theta) = (-1)^n \cos\theta$
$\tan(n\pi+\theta) = \tan\theta$

(f) $\sin(n\pi-\theta) = (-1)^{n+1}\sin\theta$
$\cos(n\pi-\theta) = (-1)^n \cos\theta$
$\tan(n\pi-\theta) = -\tan\theta$.

(g) $\sin\left(\frac{n\pi}{2}+\theta\right) = (-1)^{\frac{n-1}{2}}\cos\theta$, n is odd
$(-1)^{\frac{n}{2}}\sin\theta$, n is even.

$\cos\left(\frac{n\pi}{2}+\theta\right) = (-1)^{\frac{n+1}{2}}\sin\theta$, n is odd
$= (-1)^{\frac{n}{2}}\cos\theta$, n is even.

$\tan\left(\frac{n\pi}{2}+\theta\right) = -\cot\theta$, n is odd
$= \tan\theta$, n is even.

3.: T-ratio's of some angle's (0°, 30°, 45°, 60°, 90°, 120°, 135°, 150°, 180°)

(i) Note that $\sin 0 = 0$, $\sin 30° = \sin\frac{\pi}{6} = \frac{1}{2}$, $\sin 45° = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$
$\sin 60° = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\sin 90° = \sin\left(\frac{\pi}{2}\right) = 1$.

(ii) To find $\cos\theta$ Apply $90-\theta$ formula for acute angles and $180-\theta$ for obtuse angles.

(iii) For other T-ratios Apply $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$
$\sec\theta = \frac{1}{\cos\theta}$, $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$.

4. ① Trigonometrical ~~Pros~~ identities:

$\boxed{\sin^2\theta + \cos^2\theta = 1}$

$\boxed{\sec^2\theta - \tan^2\theta = 1}$

$\boxed{\operatorname{cosec}^2\theta - \cot^2\theta = 1}$

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② Reciprocal Relation:

(a) $\langle \sin\theta, \operatorname{cosec}\theta \rangle$

$\sin\theta \times \operatorname{cosec}\theta = 1$ or $\sin\theta = \frac{1}{\operatorname{cosec}\theta}$ or $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$

(b) $\langle \cos\theta, \sec\theta \rangle$

$\cos\theta \times \sec\theta = 1$ or $\cos\theta = \frac{1}{\sec\theta}$ or $\sec\theta = \frac{1}{\cos\theta}$

(c) $\langle \tan\theta, \cot\theta \rangle$

$\tan\theta \times \cot\theta = 1$ or $\tan\theta = \frac{1}{\cot\theta}$ or $\cot\theta = \frac{1}{\tan\theta}$

(3) Division Relation:

(a) $\tan\theta = \frac{\sin\theta}{\cos\theta}$ (b) $\cot\theta = \frac{\cos\theta}{\sin\theta}$

④ PRODUCT OF T-ratios:

$\sin\theta \times \sin\theta \times \sin\theta$ ------ p times $= (\sin\theta)^p$
$= \sin^p\theta$. Note that $\sin^p\theta \neq \sin(p\theta)$.

similarly for other T-ratios.

⑤ Sum / Difference of T-ratios:

$\sin\theta + \sin\theta +$ -------- p times $= p\sin\theta$

similarly, for other T-ratios.

⑥ Quotient of two T-ratios:

$$\frac{\sin^p\theta}{\sin^q\theta} = \sin^{p-q}\theta$$

Problems:

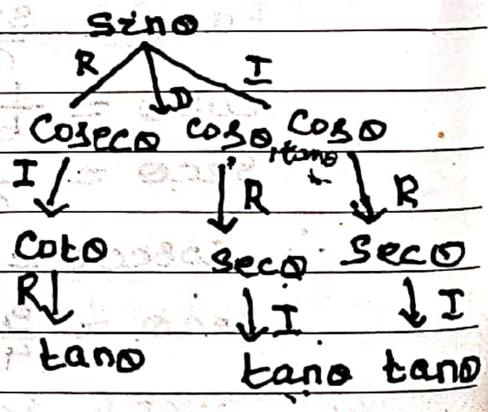
① Convert $\sin\theta$ into $\tan\theta$.

Method I Soln $\sin\theta = \frac{1}{\operatorname{cosec}\theta}$

$$= \frac{1}{\sqrt{1+\cot^2\theta}}$$

$$= \frac{1}{\sqrt{1+\frac{1}{\tan^2\theta}}}$$

$$= \frac{1}{\sqrt{\frac{1+\tan^2\theta}{\tan^2\theta}}} = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$$



(2) Express $\cos 1400^\circ$ in terms of an acute angle.

Soln $\cos 1400^\circ = \cos(7\pi + 140) = (-1)^7 \cos 140 = -\cos 140$
$= -\cos(\frac{\pi}{2} + 50) = -(-\sin 50) = \sin 50^\circ$ (Ans)

(3) Find $\sin 1^\circ \cdot \sin 2^\circ \cdots\cdots \sin 200^\circ$

soln: $\sin 1^\circ \cdot \sin 2^\circ \cdots\cdots \sin 200^\circ$
$= \sin 1^\circ \cdot \sin 2^\circ \cdots\cdots \sin 180^\circ \cdot \sin 181^\circ \cdots\cdots \sin 200^\circ$
$= 0$ $(\because \sin 180^\circ = 0)$.

(4) Find $\dfrac{\cot(\pi+\theta)\cdot\cot(\frac{\pi}{2}+\theta)\cdot\cos(4\pi-\theta)}{\tan(\pi/2-\theta)\cdot\operatorname{cosec}(\pi-\theta)\cdot\sin(-\theta)}$.

Soln: $\dfrac{\cot(\pi+\theta)\cdot\cot(\frac{\pi}{2}+\theta)\cdot\cos(4\pi-\theta)}{\tan(\pi/2-\theta)\cdot\operatorname{cosec}(\pi-\theta)\cdot\sin(-\theta)}$

$= \dfrac{(\cot\theta)\times(-\tan\theta)\times(-1)^4\cos\theta}{\cot\theta\times\operatorname{cosec}\theta\times(-\sin\theta)} = \dfrac{(-1)\times\cos\theta}{\cot\theta\times(-1)} = \cos\theta\times\tan\theta$

$= \cos\theta \times \dfrac{\sin\theta}{\cos\theta} = \sin\theta$ (Ans)

(5) If $\sin\theta = \frac{3}{5}$, $0<\theta<\frac{\pi}{2}$ then find other T-ratios.

Soln: $\sin\theta = \frac{3}{5}$

$\operatorname{cosec}\theta = \dfrac{1}{\sin\theta} = \dfrac{1}{(3/5)} = \dfrac{5}{3}$

$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-(3/5)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{25-9}{25}} = \sqrt{\frac{16}{25}}$

$\Rightarrow \cos\theta = \pm\frac{4}{5} \Rightarrow \cos\theta = \frac{4}{5}$ $\because 0<\theta<\frac{\pi}{2}, \cos\theta>0$.

$\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{3/5}{4/5} = \dfrac{3}{4}$

$\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{4}{3}$

$\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{1}{(4/5)} = \dfrac{5}{4}$

$\therefore \operatorname{cosec}\theta = \frac{5}{3}$, $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$, $\cot\theta = \frac{4}{3}$,
$\sec\theta = \frac{5}{4}$.

HOME TASK:

(6) Convert $\sec\theta$ into $\operatorname{cosec}\theta$

(7) Express $\tan 145^\circ$ in terms of an acute angle.

(8) Find the value of $\tan 1020^\circ$.

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(9) Express $\cos\theta$ into $\operatorname{cosec}\theta$

(10) Find $\dfrac{\cos(180-A)\times\cot(90+A)\times\cos(-A)}{\tan(180+A)\times\tan(270+A)\times\sin(360-A)}$

(11) Find $\cos 1 \cdot \cos 2^\circ \cdot \cos 3^\circ \cdots\cdots \cos 100^\circ$.

(12) Find $\sin(-330)\cdot\cos 300 + \cos(-390)\times\sin 420$

(I) T-Ratios of Compound Angle: A+B, A−B

(i) $\sin(A+B) = \sin A\cdot\cos B + \cos A\cdot\sin B$.

$\sin(A-B) = \sin A\cdot\cos B - \cos A\cdot\sin B$.

(ii) $\cos(A+B) = \cos A\cdot\cos B - \sin A\cdot\sin B$

$\cos(A-B) = \cos A\cdot\cos B + \sin A\cdot\sin B$.

(iii) $\tan(A+B) = \dfrac{\tan A+\tan B}{1-\tan A\cdot\tan B}$

$\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A\cdot\tan B}$

(iv) $\cot(A+B) = \dfrac{\cot A\cdot\cot B - 1}{\cot B + \cot A}$

$\cot(A-B) = \dfrac{\cot A\cdot\cot B + 1}{\cot B - \cot A}$.

PROBLEMS:

(13) Show that $\tan 15^\circ = 2-\sqrt{3}$.

Sol^n $\tan 15^\circ = \tan(60-45) = \dfrac{\tan 60 - \tan 45}{1+\tan 60^\circ\cdot\tan 45} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

$= \dfrac{(\sqrt{3}-1)^2}{(\sqrt{3}+1)(\sqrt{3}-1)} = \dfrac{3+1-2\sqrt{3}}{(\sqrt{3})^2-(1)^2} = \dfrac{4-2\sqrt{3}}{3-1} = \dfrac{2(2-\sqrt{3})}{2}$

$= 2-\sqrt{3}$ $\square$

(14) If $\sin A = \dfrac{1}{\sqrt{10}}$, $\sin B = \dfrac{1}{\sqrt{5}}$ then show that $A+B = \dfrac{\pi}{4}$

Sol^n $\sin A = \dfrac{1}{\sqrt{10}}$, $\sin B = \dfrac{1}{\sqrt{5}}$

$\cos A = \sqrt{1-\sin^2 A} = \sqrt{1-\left(\dfrac{1}{\sqrt{10}}\right)^2} = \sqrt{1-\dfrac{1}{10}} = \sqrt{\dfrac{10-1}{10}}$

$= \dfrac{3}{\sqrt{10}}$

$\cos B = \sqrt{1-\sin^2 B} = \sqrt{1-\left(\dfrac{1}{\sqrt{5}}\right)^2} = \sqrt{1-\dfrac{1}{5}} = \sqrt{\dfrac{5-1}{5}}$

$= \dfrac{2}{\sqrt{5}}$

Now $\sin(A+B) = \sin A.\cos B + \cos A.\sin B = \frac{1}{\sqrt{10}}.\frac{2}{\sqrt{5}} + \frac{3}{\sqrt{10}}.\frac{1}{\sqrt{5}}$

$= \frac{2+3}{\sqrt{5}.\sqrt{10}} = \frac{5}{\sqrt{5}.\sqrt{5}.\sqrt{2}} = \frac{5}{5.\sqrt{2}} = \frac{1}{\sqrt{2}}$

$\therefore \sin(A+B) = \frac{1}{\sqrt{2}} \Rightarrow \sin(A+B) = \sin\frac{\pi}{4} \Rightarrow A+B = \frac{\pi}{4}$ □.

15) If $A+B = 45°$ then show that $(1+\tan A)(1+\tan B) = 2$.

Soln $A+B = 45 \Rightarrow \tan(A+B) = \tan 45° \Rightarrow \tan(A+B) = 1$

$\Rightarrow \frac{\tan A + \tan B}{1-\tan A.\tan B} = 1 \Rightarrow \tan A + \tan B = 1 - \tan A.\tan B$

$\Rightarrow \tan A + \tan B + \tan A.\tan B = 1 \Rightarrow 1 + \tan A + \tan B + \tan A.\tan B = 1+1$

$\Rightarrow 1(1+\tan A) + \tan B(1+\tan A) = 2 \Rightarrow (1+\tan A)(1+\tan B) = 2$ □.

16) Show that $\frac{\cos 9° + \sin 9°}{\cos 9° - \sin 9°} = \tan 54°$.

Method-1 $54° = 45° + 9° \Rightarrow \tan 54° = \tan(45° + 9°)$

$\Rightarrow \tan 54° = \frac{\tan 45° + \tan 9}{1 - \tan 45.\tan 9} = \frac{1+\tan 9}{1-\tan 9} = \frac{1 + \frac{\sin 9}{\cos 9}}{1 - \frac{\sin 9}{\cos 9}}$

$= \frac{\cos 9 + \sin 9}{\cos 9} \Big/ \frac{\cos 9 - \sin 9}{\cos 9} = \frac{\cos 9 + \sin 9}{\cos 9} \times \frac{\cos 9}{\cos 9 - \sin 9}$

$= \frac{\cos 9 + \sin 9}{\cos 9 - \sin 9}$ □

Method-2 $LHS = \frac{\cos 9 + \sin 9}{\cos 9 - \sin 9} = \frac{1+\tan 9}{1-\tan 9}$ (Dividing N^r & D^r by $\cos 9°$)

$= \cancel{\frac{\tan 45 + \tan 9}{\tan 45 - \tan 9}} = \frac{\tan 45 + \tan 9}{1 - \tan 45.\tan 9} = \tan(45+9) = \tan 54°$

$= RHS$.

17) In Any triangle ABC If $\cos A = \cos B.\cos C$ then show that $\tan B + \tan C = 2\tan A$.

Soln In Any triangle ABC $A+B+C = \pi$ —— ①

$\cos A = \cos B.\cos C$ —— ②

From ① $A+B+C = \pi \Rightarrow B+C = \pi - A$

~~$\Rightarrow \tan(B+C) = \tan(\pi - A) \Rightarrow \tan B +$~~

$\Rightarrow \sin(B+C) = \sin(\pi - A)$

$\Rightarrow \sin B.\cos C + \cos B.\sin C = \sin A$

$\Rightarrow \frac{\sin B.\cos C + \cos B.\sin C}{\cos A} = \frac{\sin A}{\cos A}$

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$$\Rightarrow \frac{\sin B.\cos C}{\cos A} + \frac{\cos B.\sin C}{\cos A} = \tan A$$

$$\Rightarrow \frac{\sin B.\cos C}{\cos B.\cos C} + \frac{\cos B.\sin C}{\cos B.\cos C} = \tan A$$

$$\Rightarrow \frac{\sin B}{\cos B} + \frac{\sin C}{\cos C} = \tan A \Rightarrow \tan B + \tan C = \tan A \ \square.$$

HOME TASK:

(18) Find the value of (a) $\cos 15^\circ$ (b) $\sin 15^\circ$ (c) $\tan 75^\circ$ (d) $\cot 75^\circ$ (e) $\sin 75^\circ$ (f) $\cos 75^\circ$ (g) $\cot 15^\circ$

(19) Show that $\tan 75 + \cot 75 = 4$

(20) Find $\dfrac{\tan 20 + \tan 25}{1 - \tan 20.\tan 25}$

(21) Find $\sin 35^\circ.\cos 25 + \cos 35^\circ.\sin 25^\circ$

(22) If $\tan A = \frac{5}{6}$, $\tan B = \frac{1}{11}$ then prove that $A+B = \frac{\pi}{4}$

(23) Show that $\dfrac{\sin(A-B)}{\cos A.\cos B} + \dfrac{\sin(B-C)}{\cos B.\cos C} + \dfrac{\sin(C-A)}{\cos C.\cos A} = 0$

(24) (a) Show that if $A+B = 45^\circ$ then $(\cot A - 1)(\cot B - 1) = 2$.

(b) Show that $(1+\tan 25)(1+\tan 20) = 2$.

(c) Show that $(\cot 23^\circ - 1)(\cot 22^\circ - 1) = 2$.

(25) If $\tan A + \tan B = p$, $\cot A + \cot B = q$ then show that $\cot(A+B) = \frac{1}{p} - \frac{1}{q}$

(26) Show that $\dfrac{\cos 18 + \sin 18}{\cos 18 - \sin 18} = \tan 63^\circ$

(27) If $A+B+C = \pi$ & $\cos A = \cos B.\cos C$ then show that $\tan B.\tan C = 2$.

(28) Show that $\tan 3\theta - \tan \theta - \tan 2\theta = \tan \theta.\tan 2\theta.\tan 3\theta$

(29) In any triangle ABC show that $\tan A + \tan B + \tan C = \tan A.\tan B.\tan C$.

1) TRANSFORMATION OF SUM / DIFFERENCE INTO PRODUCT

(i) $\sin(A+B)+\sin(A-B)=2\sin A.\cos B$

(ii) $\sin(A+B)-\sin(A-B)=2\cos A.\sin B$

(iii) $\cos(A+B)+\cos(A-B)=2\cos A.\cos B$

(iv) $\cos(A+B)-\cos(A-B)=-2\sin A.\sin B.$

(v) $\sin C+\sin D=2\sin\left(\frac{C+D}{2}\right).\cos\left(\frac{C-D}{2}\right)$

(vi) $\sin C-\sin D=2\cos\left(\frac{C+D}{2}\right).\sin\left(\frac{C-D}{2}\right)$

(vii) $\cos C+\cos D=2\cos\left(\frac{C+D}{2}\right).\cos\left(\frac{C-D}{2}\right)$

(viii) $\cos C-\cos D=-2\sin\left(\frac{C+D}{2}\right).\sin\left(\frac{C-D}{2}\right)$

(ix) $\sin(A+B).\sin(A-B)=\sin^2A-\sin^2B=\cos^2B-\cos^2A$

(x) $\cos(A+B).\cos(A-B)=\cos^2A-\sin^2B=\cos^2B-\sin^2A.$

2) T-Ratios of Multiple Angle: (2A, 3A).

(i) $\sin 2A=2\sin A.\cos A=\dfrac{2\tan A}{1+\tan^2A}$

(ii) $\cos 2A=\cos^2A-\sin^2A=2\cos^2A-1=1-2\sin^2A=\dfrac{1-\tan^2A}{1+\tan^2A}$

(iii) $\tan 2A=\dfrac{2\tan A}{1-\tan^2A}$

(iv) $\cot 2A=\dfrac{\cot^2A-1}{2\cot A}$

(v) $\sin 3A=3\sin A-4\sin^3A$

(vi) $\cos 3A=4\cos^3A-3\cos A$

(vii) $\tan 3A=\dfrac{3\tan A-\tan^3A}{1-3\tan^2A}$

(viii) $\cot 3A=\dfrac{\cot^3A-3\cot A}{3\cot^2A-1}$

(L) T-ratios of sub multiple angles.

(a) Angle A in terms of A/2:

(i) $\sin A = 2\sin(\frac{A}{2})\cdot\cos(\frac{A}{2}) = \frac{2\tan(\frac{A}{2})}{1+\tan^2(\frac{A}{2})}$

(ii) $\cos A = \cos^2(A/2)-\sin^2(A/2) = 2\cos^2(A/2)-1 = 1-2\sin^2(A/2) = \frac{1-\tan^2(A/2)}{1+\tan^2(A/2)}$

(iii) $\tan A = \frac{2\tan(A/2)}{1-\tan^2(A/2)}$

(b) Angle A in terms of A/3

(i) $\sin A = 3\sin(A/3) - 4\sin^3(A/3)$

(ii) $\cos A = 4\cos^3(A/3) - 3\cos(A/3)$

(iii) $\tan A = \frac{3\tan(A/3) - \tan^3(A/3)}{1-3\tan^2(A/3)}$

(iv) $\cot A = \frac{\cot^3(A/3) - 3\cot(A/3)}{3\cot^2(A/3) - 1}$

(c) Angle A/2 in terms of A

(i) $2\cos^2 A = 1+\cos 2A \Rightarrow 2\cos^2(A/2) = 1+\cos A$

(ii) $2\sin^2 A = 1-\cos 2A \Rightarrow 2\sin^2(A/2) = 1-\cos A$.

(iii) $\sin(\frac{A}{2}) + \cos(\frac{A}{2}) = \sqrt{1+\sin A}$ if $-\frac{\pi}{4} < \frac{A}{2} < \frac{3\pi}{4}$
$= -\sqrt{1+\sin A}$ otherwise } —(i)

$\sin(\frac{A}{2}) - \cos(\frac{A}{2}) = \sqrt{1-\sin A}$ if $\frac{\pi}{4} < \frac{A}{2} < \frac{5\pi}{4}$
$= -\sqrt{1-\sin A}$ otherwise } —(ii)

Adding (i) & (ii) we get $\sin(\frac{A}{2})$ and substracting (i) and (ii) we get $\cos(\frac{A}{2})$

(iv) $\tan(\frac{A}{2}) = \frac{1-\cos A}{\sin A} = \frac{\sin A}{1+\cos A}$

(v) $\cot(\frac{A}{2}) = \frac{1+\cos A}{\sin A} = \frac{\sin A}{1-\cos A}$

NOTE: (i) $4\sin A\cdot\sin(60-A)\cdot\sin(60+A) = \sin 3A$ ✓

(ii) $4\cos A\cdot\cos(60-A)\cdot\cos(60+A) = \cos 3A$ ✓

(iii) $\sin 15^\circ = \cos 75^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$, $\cos 15^\circ = \sin 75^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$

$\tan 15^\circ = \cot 75^\circ = 2-\sqrt{3}$, $\cot 15^\circ = \tan 75^\circ = 2+\sqrt{3}$

(iv) $\sin 18^\circ = \cos 72^\circ = \frac{\sqrt{5}-1}{4}$, $\cos 18 = \sin 72^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}$

$\sin 36^\circ = \cos 54^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}$, $\cos 36 = \sin 54^\circ = \frac{\sqrt{5}+1}{4}$.

Maximum and minimum value of $a\sin\theta + b\cos\theta$.

(i) $\max.(\sin\theta) = 1$ for $\theta = \frac{\pi}{2}$ | $\max\cos\theta = 1$ for $\theta = 0^\circ$

$\min.(\sin\theta) = -1$ for $\theta = \frac{3\pi}{2}$ | $\min\cos\theta = -1$ for $\theta = \pi$

(ii) To find maximum and minimum value of $a\sin\theta + b\cos\theta$,
put $a = r\cos\alpha$, $b = r\sin\alpha$ so that $r = \sqrt{a^2+b^2}$
then $a\sin\theta + b\cos\theta = r\cos\alpha\cdot\sin\theta + r\sin\alpha\cos\theta$

$= r\sin(\theta+\alpha)$ ——— (1)

since $\max\sin(\theta+\alpha) = 1 \Rightarrow \max\{a\sin\theta + b\cos\theta\} = r = \sqrt{a^2+b^2}$

since $\min.\sin(\theta+\alpha) = -1 \Rightarrow \min.\{a\sin\theta + b\cos\theta\} = -r = -\sqrt{a^2+b^2}$

PROBLEMS

(30) Show that $\sin(A+B)\cdot\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

Soln $\sin(A+B)\cdot\sin(A-B)$ ——— (i)

$= (\sin A\cdot\cos B + \cos A\cdot\sin B)(\sin A\cdot\cos B - \cos A\cdot\sin B)$

$= (\sin A\cos B)^2 - (\cos A\cdot\sin B)^2 = \sin^2 A\cdot\cos^2 B - \cos^2 A\cdot\sin^2 B$

$= \sin^2 A(1-\sin^2 B) - \sin^2 B(1-\sin^2 A)$

$= \sin^2 A - \sin^2 A\cdot\sin^2 B - \sin^2 B + \sin^2 A\cdot\sin^2 B$

$= \sin^2 A - \sin^2 B$ ——— (ii)

$= (1-\cos^2 A) - (1-\cos^2 B) = 1 - \cos^2 A - 1 + \cos^2 B = \cos^2 B - \cos^2 A$ ——— (iii)

From (i), (ii), (iii) we have

$\sin(A+B)\cdot\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$ □

(31) Prove that $4\sin A \cdot \sin(60-A)\cdot \sin(60+A) = \sin 3A$.

$LHS = 4\sin A \cdot \sin(60-A)\cdot\sin(60+A)$

$= 4\sin A\{\sin^2 60 - \sin^2 A\}$ $\qquad \because \sin(A+B)\cdot\sin(A-B) = \sin^2 A - \sin^2 B$.

$= 4\sin A\{(\frac{\sqrt{3}}{2})^2 - \sin^2 A\}$

$= 4\sin A\{\frac{3}{4} - \sin^2 A\} = 4\sin A\{\frac{3-4\sin^2 A}{4}\}$

$= \sin A\{3-4\sin^2 A\} = 3\sin A - 4\sin^3 A = \sin 3A = RHS$

(32) Find the value of $\sin 18°, \cos 18°, \sin 36°, \cos 36, \sin 54, \cos 54°, \sin 72°, \cos 72°$.

Soln Let $A = 18° \Rightarrow 5A = 90° \Rightarrow 2A + 3A = 90 \Rightarrow 2A = 90-3A$

$\Rightarrow \sin 2A = \sin(90-3A) \Rightarrow \sin 2A = \cos 3A$

$\Rightarrow 2\sin A\cdot\cos A = 4\cos^3 A - 3\cos A$

$\Rightarrow 2\sin A\cdot\cos A = \cos A(4\cos^2 A - 3)$

$\Rightarrow 2\sin A = 4\cos^2 A - 3$ $\qquad (\because A = 18° \Rightarrow \cos A \neq 0)$

$\Rightarrow 2\sin A = 4(1-\sin^2 A) - 3 = 4 - 4\sin^2 A - 3 = 1 - 4\sin^2 A$

$\Rightarrow$ ~~$4\sin^2 A + 1$~~ $2\sin A + 4\sin^2 A - 1 = 0$

$\Rightarrow 4\sin^2 A + 2\sin A - 1 = 0$

$\Rightarrow \sin A = \frac{-2 \pm \sqrt{(2)^2 - 4\cdot 4\cdot(-1)}}{2\times 4} = \frac{-2\pm\sqrt{4+16}}{8} = \frac{-2\pm\sqrt{20}}{8}$

$\Rightarrow \sin A = \frac{-2\pm 2\sqrt{5}}{8} = \frac{-1\pm\sqrt{5}}{4} = \frac{-1+\sqrt{5}}{4}$ or $\frac{-1-\sqrt{5}}{4}$

$\Rightarrow \sin A = \frac{\sqrt{5}-1}{4}$ $\qquad (\because A = 18° \Rightarrow \sin A > 0)$

$\Rightarrow \sin 18° = \frac{\sqrt{5}-1}{4}$

Now $\cos 18 = \pm\sqrt{1-\sin^2 18} = \pm\sqrt{1-(\frac{\sqrt{5}-1}{4})^2}$

$= \pm\sqrt{1 - \frac{5+1-2\sqrt{5}}{16}} = \pm\sqrt{1-\frac{6-2\sqrt{5}}{16}}$

$= \pm\sqrt{1-\frac{3-\sqrt{5}}{8}} = \pm\sqrt{\frac{8-3+\sqrt{5}}{8}} = \pm\sqrt{\frac{5+\sqrt{5}}{8}}$

$= \pm\sqrt{\frac{10+2\sqrt{5}}{16}} = \frac{\sqrt{10+2\sqrt{5}}}{4}$ $\qquad (\because \cos 18° > 0)$

Now $\cos 36° = \cos(2\times 18) = 1-2\sin^2 18 = 1-2\times\left(\frac{\sqrt5-1}{4}\right)^2$

$= 1-2\times\frac{5+1-2\sqrt5}{16} = 1-\frac{6-2\sqrt5}{8} = \frac{8-6+2\sqrt5}{8}$

$= \frac{2+2\sqrt5}{8} = \frac{\sqrt5+1}{4}$

$\cancel{\sin 36° = \sin(2\times18) = 2\sin 18\cdot\cos 18 = 2\times\frac{\sqrt5-1}{4}\times\frac{\sqrt{10+2\sqrt5}}{4}}$

$\sin 36 = \pm\sqrt{1-\cos^2 36} = \pm\sqrt{1-\left(\frac{\sqrt5+1}{4}\right)^2} = \pm\sqrt{1-\frac{5+1+2\sqrt5}{16}}$

$= \pm\sqrt{\frac{16-6-2\sqrt5}{16}} = \frac{\sqrt{10-2\sqrt5}}{4}$ $(\because \sin 36° > 0)$

$\sin 72° = \sin(90-18) = \cos 18 = \frac{\sqrt{10+2\sqrt5}}{4}$

$\cos 72 = \cos(90-18) = \sin 18 = \frac{\sqrt5-1}{4}$

$\sin 54° = \sin(90-36) = \cos 36° = \frac{\sqrt5+1}{4}$

$\cos 54° = \cos(90-36) = \sin 36° = \frac{\sqrt{10+2\sqrt5}}{4}$

33) Show that the equation $\cos\theta = a+\frac{1}{a}$ doesnot have a solution if $a\neq 0$ is real.

Soln $a\neq 0,\ a\in\mathbb{R}$.

Now $\left(a+\frac{1}{a}\right)^2 = \left(a-\frac{1}{a}\right)^2+4\cdot a\cdot\frac{1}{a} = \left(a-\frac{1}{a}\right)^2+4$

But since $\left(a-\frac{1}{a}\right)^2 \geq 0$

$\Rightarrow \left(a+\frac{1}{a}\right)^2 \geq 0+4 \Rightarrow \left(a+\frac{1}{a}\right)^2 \geq 4$

$\Rightarrow x^2 \geq 4 \Rightarrow x^2-4 \geq 0$ (put $a+\frac{1}{a} = x = \cos\theta$)

$\Rightarrow (x+2)(x-2) \geq 0 \Rightarrow x+2\geq 0$ and $x-2\geq 0$

or $(x+2)\leq 0$ and $(x-2)\leq 0$

$\Rightarrow x\geq -2$ and $x\geq 2$ or $x\leq -2$ and $x\leq 2$.

$\Rightarrow x\geq 2$ or $x\leq -2$

$\Rightarrow \cos\theta \geq 2$ or $\cos\theta\leq -2$ which is impossible for $-1\leq\cos\theta\leq 1$. Hence the result.

34) If $A = \cos^2\theta + \sin^4\theta$, then prove that for all values of θ, $\frac{3}{4} \leq A \leq 1$

Soln $A = \cos^2\theta + \sin^4\theta = 1 - \sin^2\theta + \sin^4\theta$

$\Rightarrow \sin^4\theta - \sin^2\theta + (1-A) = 0$

$\Rightarrow x^2 - x + (1-A) = 0$ (put $x = \sin^2\theta$)

$\Rightarrow x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4\cdot1\cdot(1-A)}}{2\times1} = \frac{1 \pm \sqrt{4A-3}}{2}$

$\Rightarrow \sin^2\theta = \frac{1 \pm \sqrt{4A-3}}{2}$ ——— (1)

Since $\sin\theta$ is real $\Rightarrow \sin^2\theta$ is real

$\Rightarrow$ Discriminant $\geq 0 \Rightarrow 4A-3 \geq 0 \Rightarrow 4A \geq 3$

$\Rightarrow A \geq \frac{3}{4} \Rightarrow \frac{3}{4} \leq A$ ——— (2)

Again since $-1 \leq \sin\theta \leq 1 \Rightarrow \sin^2\theta \leq 1$

$\Rightarrow \frac{1 \pm \sqrt{4A-3}}{2} \leq 1 \Rightarrow 1 \pm \sqrt{4A-3} \leq 2 \Rightarrow \pm\sqrt{4A-3} \leq 1$

$\Rightarrow 4A - 3 \leq 1 \Rightarrow 4A \leq 4 \Rightarrow A \leq 1$ ——— (3)

Combining (2) & (3) we have $\frac{3}{4} \leq A \leq 1$ $\square$.

35) Show that

$2^n \cdot \cos\theta \cdot \cos 2\theta \cdot \cos 2^2\theta \cdot \ldots \cos 2^{n-1}\theta = 1$ if $\theta = \frac{\pi}{2^n+1}$

Soln $\theta = \frac{\pi}{2^n+1} \Rightarrow (2^n+1)\theta = \pi \Rightarrow 2^n\theta + \theta = \pi$

$\Rightarrow 2^n \cdot \theta = \pi - \theta$ ——— (1)

Now LHS $= 2^n \cdot \cos\theta \cdot \cos 2\theta \cdot \cos 2^2\theta \cdot \cos 2^3\theta \ldots \cos 2^{n-1}\theta$

$= \frac{2^{n-1} \cdot 2\sin\theta \cdot \cos\theta \cdot \cos 2\theta \cdot \cos 2^2\theta \cdot \cos 2^3\theta \ldots \cos 2^{n-1}\theta}{\sin\theta}$

$= \frac{2^{n-1} \cdot \sin 2\theta \cdot \cos 2\theta \cdot \cos 2^2\theta \cdot \cos 2^3\theta \ldots \cos 2^{n-1}\theta}{\sin\theta}$

$= \frac{2^{n-2} \cdot 2\sin 2\theta \cdot \cos 2\theta \cdot \cos 2^2\theta \cdot \cos 2^3\theta \ldots \cos 2^{n-1}\theta}{\sin\theta}$

$$= \frac{2^{n-2}\,\sin 2^2\theta \cdot \cos 2^2\theta \cdot \cos 2^3\theta \cdots\cdots \cos 2^{n-1}\theta}{\sin\theta}$$

$$= \frac{2^{n-3}\, 2\sin 2^2\theta \cdot \cos 2^2\theta \cdot \cos 2^3\theta \cdots\cdots \cos 2^{n-1}\theta}{\sin\theta}$$

$$= \frac{2^{n-3}\,\sin 2^3\theta \cdot \cos 2^3\theta \cdots\cdots \cos 2^{n-1}\theta}{\sin\theta}$$

$$\vdots$$

$$= \frac{2^{n-(n-1)}\,\sin 2^{n-1}\theta \cdot \cos 2^{n-1}\theta}{\sin\theta} = \frac{2\sin 2^{n-1}\theta \cdot \cos 2^{n-1}\theta}{\sin\theta}$$

$$= \frac{\sin(2\times 2^{n-1}\theta)}{\sin\theta} = \frac{\sin(2^n\theta)}{\sin\theta} = \frac{\sin(\pi-\theta)}{\sin\theta}$$

$$= \frac{\sin\theta}{\sin\theta} = 1 = \text{RHS} \quad \square.$$

(35) Prove that $\sqrt{\frac{1+\sin A}{1-\sin A}} = \tan\left(\frac{\pi}{4}+\frac{A}{2}\right)$

Solⁿ LHS $= \sqrt{\frac{1+\sin A}{1-\sin A}} = \sqrt{\frac{1+2\sin(\frac{A}{2})\cdot\cos(\frac{A}{2})}{1-2\sin(\frac{A}{2})\cdot\cos(\frac{A}{2})}} =$

$$= \sqrt{\frac{\cos^2(\frac{A}{2})+\sin^2(\frac{A}{2})+2\sin(\frac{A}{2})\cdot\cos(\frac{A}{2})}{\cos^2(\frac{A}{2})+\cos^2(\frac{A}{2})-2\sin(\frac{A}{2})\cdot\cos(\frac{A}{2})}}$$

$$= \sqrt{\frac{[\cos(A/2)+\sin(A/2)]^2}{[\cos(A/2)-\sin(A/2)]^2}} = \frac{\cos(A/2)+\sin(\frac{A}{2})}{\cos(A/2)-\sin(\frac{A}{2})}$$

$$= \frac{1+\tan(\frac{A}{2})}{1-\tan(\frac{A}{2})}$$ Dividing N^r & D^r by $\cos(A/2)$

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$$= \frac{\tan(\frac{\pi}{4}) + \tan(\frac{A}{2})}{1 - \tan(\frac{\pi}{4}).\tan(\frac{A}{2})} = \tan\left(\frac{A}{2} + \frac{\pi}{4}\right) = RHS \ \square.$$

(36) Show that $\tan(\frac{A}{2}) = \frac{1-\cos A}{\sin A} = \frac{\sin A}{1+\cos A}$

Soln $\tan(\frac{A}{2}) = \frac{\sin(A/2)}{\cos(A/2)} = \cancel{\frac{2\sin(A/2)\sin(A/2)}{2\cos A}}$

$$= \frac{2\sin^2(A/2)}{2\sin(A/2).\cos(A/2)} = \frac{1-\cos(2\times A/2)}{\sin(2\times A/2)} = \frac{1-\cos A}{\sin A} \text{ ---(i)}$$

$$= \frac{(1-\cos A)(1+\cos A)}{\sin A(1+\cos A)} = \frac{1-\cos^2 A}{\sin A(1+\cos A)} = \frac{\sin^2 A}{\sin A(1+\cos A)}$$

$$= \frac{\sin A}{1+\cos A} \text{ ---(ii)}$$

From (i) & (ii) we have $\tan(\frac{A}{2}) = \frac{1-\cos A}{\sin A} = \frac{\sin A}{1+\cos A}$ $\square$.

(37) Find maximum and minimum value of $8\cos x - 15\sin x - 12$

Soln Let $P(x) = 8\cos x - 15\sin x - 12$ ---①

put $8 = r\sin\alpha$ & $15 = r\cos\alpha$

$r = \sqrt{8^2+15^2} = \sqrt{64+225} = \sqrt{289} = 17$

Now $P(x) = 8\cos x - 15\sin x - 12$

$= r\sin\alpha.\cos x - r\cos\alpha.\sin x - 12$

$= r\sin(\alpha - x) - 12 = 17\sin(\alpha - x) - 12$ ---②

$\because$ max. $\sin(\alpha - x) = 1 \Rightarrow$ max. $P(x) = 17\times 1 - 12 = 5$

$\because$ min. $\sin(\alpha - x) = -1 \Rightarrow$ min $P(x) = 17\times(-1) - 12 = -29$.

$\therefore$ max$\{8\cos x - 15\sin x - 12\} = 5$ & min$\{8\cos x - 15\sin x - 12\} = -29$

(38) Find maximum and minimum value of $\sin\theta.\cos\theta$. and hence find for what values of θ it is maximum and minimum.

Soln Let $P(\theta) = \sin\theta.\cos\theta = \frac{1}{2}(2\sin\theta\cos\theta) = \frac{1}{2}\sin 2\theta$ ---①

$\because$ max. $\sin 2\theta = 1 \Rightarrow$ max $P(\theta) = \frac{1}{2}\times 1 = \frac{1}{2}$.

Also it is maximum for $2\theta = \frac{\pi}{2}$ i.e $\theta = \frac{\pi}{4}$

Again $\because$ min$(\sin 2\theta) = -1 \Rightarrow$ min $P(\theta) = \frac{1}{2}\times(-1) = -\frac{1}{2}$.

Also it is minimum for $2\theta = \frac{3\pi}{2}$ i.e $\theta = \frac{3\pi}{4}$

(39) Show that $\cos 20^\circ \cdot \cos 40^\circ \cdot \cos 60^\circ \cdot \cos 80^\circ = \frac{1}{16}$

Soln LHS $= \cos 20^\circ \cdot \cos 40^\circ \cdot \cos 60^\circ \cdot \cos 80^\circ$

$= \frac{1}{2} \cdot \cos 20^\circ \cdot \cos(60+20) \cdot \cos(60-20)$ $\because \cos 60^\circ = \frac{1}{2}$

$= \frac{1}{8} \times 4 \cos 20 \cdot \cos(60+20) \cos(60-20)$

$= \frac{1}{8} \times 4 \cos A \cdot \cos(60+A) \cdot \cos(60-A)$ (put $A = 20^\circ$)

$= \frac{1}{8} \times \cos 3A = \frac{1}{8} \times \cos(3\times 20) = \frac{1}{8} \cos 60 = \frac{1}{8} \times \frac{1}{2}$

$= \frac{1}{16}$ = RHS □

(40) If $\sin A = K \sin B$ then prove that $\tan \frac{1}{2}(A-B) = \frac{K-1}{K+1} \tan \frac{1}{2}(A+B)$.

Soln Given that $\sin A = K \sin B \Rightarrow \frac{\sin A}{\sin B} = K \Rightarrow \frac{\sin A}{\sin B} = \frac{K}{1}$

$\Rightarrow \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{K+1}{K-1}$ (by ~~Dividendo~~ componendo & Dividendo)

$\Rightarrow \frac{2 \sin\left(\frac{A+B}{2}\right) \cdot \cos\left(\frac{A-B}{2}\right)}{2 \cos\left(\frac{A+B}{2}\right) \cdot \sin\left(\frac{A-B}{2}\right)} = \frac{K+1}{K-1}$

$\Rightarrow \tan\left(\frac{A+B}{2}\right) \cdot \cot\left(\frac{A-B}{2}\right) = \frac{K+1}{K-1}$

$\Rightarrow \frac{\tan\left(\frac{A+B}{2}\right)}{\tan\left(\frac{A-B}{2}\right)} = \frac{K+1}{K-1} \Rightarrow (K+1) \tan\left(\frac{A-B}{2}\right) = (K-1) \tan\left(\frac{A+B}{2}\right)$

$\Rightarrow \tan \frac{1}{2}(A-B) = \frac{K-1}{K+1} \tan \frac{1}{2}(A+B)$ □

(41) If $A+B+C = \pi$ then prove that $\sin 2A + \sin 2B - \sin 2C = 4 \cos A \cdot \cos B \cdot \sin C$.

Soln $A+B+C = \pi$ ——— ①

LHS $= \sin 2A + \sin 2B - \sin 2C$

$= 2 \sin\left(\frac{2A+2B}{2}\right) \cdot \cos\left(\frac{2A-2B}{2}\right) - \sin 2C$

$= 2\sin(A+B)\cdot\cos(A-B) - \sin 2C$

$= 2\sin(\pi - C)\cdot\cos(A-B) - \sin 2C$ $\left(\because A+B+C=\pi \Rightarrow A+B = \pi - C\right)$

$= 2\sin C\cdot\cos(A-B) - 2\sin C\cdot\cos C$

$= 2\sin C\{\cos(A-B) - \cos C\}$

$= 2\sin C\{\cos(A-B) - \cos[\pi-(A+B)]\}$

$= 2\sin C\{\cos(A-B) + \cos(A+B)\}$

$= 2\sin C\cdot 2\cos A\cdot\cos B = 4\cos A\cdot\cos B\cdot\sin C.$

(42) Show that $\cos^2 A + \cos^2 B + 2\cos A.\cos B.\cos C = \sin^2 C$ If $A+B+C=\pi$

Soln $A+B+C=\pi$ —— (1)

LHS $= \cos^2 A + \cos^2 B + 2\cos A.\cos B.\cos C$

$= \cos^2 A + \cos^2 B + \cos A.\{\cos(B+C) + \cos(B-C)\}$

$= \cos^2 A + \cos^2 B + \cos A\{\cos(\pi - A) + \cos(B-C)\}$ using (1)

$= \cos^2 A + \cos^2 B + \cos A\{-\cos A + \cos(B-C)\}$

$= \cos^2 A + \cos^2 B - \cos^2 A + \cos A.\cos(B-C)$

$= \cos^2 B + \cos A.\cos(B-C) = \cos^2 B + \cos[\pi - (B+C)].\cos(B-C)$

$= \cos^2 B - \cos(B+C).\cos(B-C)$

$= \cos^2 B - \{\cos^2 B - \sin^2 C\} = \cos^2 B - \cos^2 B + \sin^2 C$

$= \sin^2 C$ = RHS □.

(43) If $A+B+C=\pi$ then prove that

$\cos\left(\frac{A}{2}\right) - \cos\left(\frac{B}{2}\right) + \cos\left(\frac{C}{2}\right) = 4\cos\left(\frac{\pi+A}{4}\right).\cos\left(\frac{\pi-B}{4}\right).\cos\left(\frac{\pi+C}{4}\right)$

Soln $A+B+C=\pi$ —— (1)

LHS $= \cos\left(\frac{A}{2}\right) - \cos\left(\frac{B}{2}\right) + \cos\left(\frac{C}{2}\right)$

$= -2\sin\left(\frac{\frac{A}{2}+\frac{B}{2}}{2}\right).\sin\left(\frac{\frac{A}{2}-\frac{B}{2}}{2}\right) + \cos\left[\frac{\pi - (A+B)}{2}\right]$ $\because A+B+C = \pi$

$= -2\sin\left(\frac{A+B}{4}\right).\sin\left(\frac{A-B}{4}\right) + \cos\left[\frac{\pi}{2} - \frac{A+B}{2}\right]$

$= -2\sin\left(\frac{A+B}{4}\right).\sin\left(\frac{A-B}{4}\right) + \sin\left(\frac{A+B}{2}\right)$

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$= -2\sin\left(\frac{A+B}{4}\right)\cdot\sin\left(\frac{A-B}{4}\right) + \sin\left(2\times\frac{A+B}{4}\right)$

$= -2\sin\left(\frac{A+B}{4}\right)\cdot\sin\left(\frac{A-B}{4}\right) + 2\sin\left(\frac{A+B}{4}\right)\cdot\cos\left(\frac{A+B}{4}\right)$

$= -2\sin\left(\frac{A+B}{4}\right)\left\{\sin\frac{A-B}{4} - \cos\frac{A+B}{4}\right\}$

~~$= -2\sin\left(\frac{\pi-C}{4}\right)\left\{\sin\left(\frac{A-B}{4}\right) + \cos\left(\frac{A-C}{4}\right)\right\}$~~

$= -2\sin\left(\frac{A+B}{4}\right)\left\{\sin\left(\frac{A-B}{4}\right) - \sin\left(\frac{\pi}{2} - \frac{A+B}{4}\right)\right\}$

$= -2\sin\left(\frac{A+B}{4}\right)\left\{\sin\left(\frac{A-B}{4}\right) - \sin\left(\frac{A+B+2C}{4}\right)\right\}$

$= -2\sin\left(\frac{A+B}{4}\right)\left\{\sin\left(\frac{A-B}{4}\right) - \sin\left(\frac{\pi+C}{4}\right)\right\}$

$= -2\sin\left(\frac{A+B}{4}\right)\left\{2\cos\left(\frac{A-B+\pi+C}{8}\right)\cdot\sin\left(\frac{A-B-\pi-C}{8}\right)\right\}$

$= -2\sin\left(\frac{A+B}{4}\right)\left\{2\cos\left(\frac{2A+2C}{8}\right)\cdot\sin\left(\frac{-2B-2C}{8}\right)\right\}$

$= +4\sin\left(\frac{A+B}{4}\right)\cdot\cos\left(\frac{A+C}{4}\right)\sin\left(\frac{B+C}{4}\right)$

$= 4\sin\left(\frac{\pi-C}{4}\right)\cdot\cos\left(\frac{\pi-B}{4}\right)\cdot\sin\left(\frac{\pi-A}{4}\right)$

$= 4\cos\left[\frac{\pi}{2} - \frac{\pi-C}{4}\right]\cdot\cos\left(\frac{\pi-B}{4}\right)\cdot\cos\left[\frac{\pi}{2} - \frac{\pi-A}{4}\right]$

$= 4\cos\left(\frac{\pi+C}{4}\right)\cos\left(\frac{\pi-B}{4}\right)\cdot\cos\left(\frac{\pi+A}{4}\right)$

$= 4\cos\left(\frac{\pi+A}{4}\right)\cdot\cos\left(\frac{\pi-B}{4}\right)\cdot\cos\left(\frac{\pi+C}{4}\right) = \text{RHS}$ □.

14) Show that $\cot 7\frac{1}{2}^\circ = \sqrt{6}+\sqrt{3}+\sqrt{2}+2$.

LHS $= \cot 7\frac{1}{2}^\circ = \cot\left(\frac{15}{2}\right)^\circ = \frac{1+\cos 15^\circ}{\sin 15^\circ}$ —— (1) $\because \cot\frac{A}{2} = \frac{1+\cos A}{\sin A}$

$\cos 15^\circ = \cos(45^\circ - 30) = \cos 45\cdot\cos 30 + \sin 45\cdot\sin 30$

$= \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\times\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$

$\sin 15^\circ = \sin(45^\circ - 30) = \sin 45\cdot\cos 30 - \cos 45^\circ\cdot\sin 30$

$= \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

From (1) $\cot 7\frac{1}{2}^\circ = \frac{1+\cos 15}{\sin 15} = \frac{1+\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}}$

$$\frac{\left(\frac{2\sqrt{2}+\sqrt{3}+1}{2\sqrt{2}}\right)}{\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)} = \frac{2\sqrt{2}+\sqrt{3}+1}{(\sqrt{3}-1)} = \frac{(2\sqrt{2}+\sqrt{3}+1)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$$

$$= \frac{2\sqrt{6}+2\sqrt{2}+3+\sqrt{3}+\sqrt{3}+1}{(\sqrt{3})^2-(1)^2} = \frac{2\sqrt{6}+2\sqrt{2}+2\sqrt{3}+4}{3-1}$$

$$= \frac{2(\sqrt{6}+\sqrt{2}+\sqrt{3}+2)}{2} = \sqrt{6}+\sqrt{3}+\sqrt{2}+2 = \text{RHS} \; \square$$

45) Show that $2\cos\left(\frac{\pi}{16}\right) = \sqrt{2+\sqrt{2+\sqrt{2}}}$.

Soln LHS $= 2\cos\left(\frac{\pi}{16}\right) = \sqrt{4\cos^2\left(\frac{\pi}{16}\right)} = \sqrt{2\cdot 2\cos^2\left(\frac{\pi}{16}\right)}$

$$= \sqrt{2\times\left(1+\cos 2\times\frac{\pi}{16}\right)} = \sqrt{2+2\cos\left(\frac{\pi}{8}\right)} = \sqrt{2+\sqrt{4\cos^2\frac{\pi}{8}}}$$

$$= \sqrt{2+\sqrt{2\cdot 2\cos^2\frac{\pi}{8}}} = \sqrt{2+\sqrt{2\left(1+\cos\left(2\times\frac{\pi}{8}\right)\right)}}$$

$$= \sqrt{2+\sqrt{2+2\cos\frac{\pi}{4}}} = \sqrt{2+\sqrt{2+2\times\frac{1}{\sqrt{2}}}}$$

$$= \sqrt{2+\sqrt{2+\sqrt{2}}} = \text{RHS} \; \square$$

46) If $(1-e)\tan^2\left(\frac{\beta}{2}\right) = (1+e)\tan^2\left(\frac{\alpha}{2}\right)$ then prove that $\cos\beta = \frac{\cos\alpha - e}{1-e\cos\alpha}$

Soln $(1-e)\tan^2\frac{\beta}{2} = (1+e)\tan^2\left(\frac{\alpha}{2}\right)$

$$\Rightarrow \tan^2\left(\frac{\beta}{2}\right) = \frac{(1+e)}{(1-e)}\tan^2\left(\frac{\alpha}{2}\right) \quad \text{—— (1)}$$

Now $\cos\beta = \cos\left(2\times\frac{\beta}{2}\right) = \frac{1-\tan^2\left(\frac{\beta}{2}\right)}{1+\tan^2\left(\frac{\beta}{2}\right)}$

$$= \frac{1-\frac{(1+e)\tan^2\left(\frac{\alpha}{2}\right)}{(1-e)}}{1+\frac{(1+e)\tan^2\left(\frac{\alpha}{2}\right)}{(1-e)}} = \frac{1-e-(1+e)\tan^2\left(\frac{\alpha}{2}\right)}{1-e+(1+e)\tan^2\left(\frac{\alpha}{2}\right)}$$

$$= \frac{1-e-\tan^2\left(\frac{\alpha}{2}\right)-e\tan^2\left(\frac{\alpha}{2}\right)}{1-e+\tan^2\left(\frac{\alpha}{2}\right)+e\tan^2\left(\frac{\alpha}{2}\right)}$$

$$\frac{\{1-\tan^2(\alpha/2)\}-e\{1+\tan^2(\frac{\alpha}{2})\}}{\{1+\tan^2(\alpha/2)\}-e\{1-\tan^2(\frac{\alpha}{2})\}} = \frac{\frac{1-\tan^2(\alpha/2)}{1+\tan^2(\alpha/2)}-e}{1-e\,\frac{1-\tan^2(\alpha/2)}{1+\tan^2(\alpha/2)}}$$

$$= \frac{\cos(2\times\alpha/2)-e}{1-e\cdot\cos(2\times\alpha/2)} = \frac{\cos\alpha-e}{1-e\cos\alpha} \quad \Box .$$

HOME TASK

(47) Show that $\cos(A+B)\cdot\cos(A-B) = \cos^2 A-\sin^2 B = \cos^2 B-\sin^2 A$.

(48) Prove that $4\cos A\cdot\cos(60+A)\cdot\cos(60-A) = \cos 3A$.

(49) Show that $\cot(\frac{A}{2}) = \frac{1+\cos A}{\sin A} = \frac{\sin A}{1-\cos A}$

(50) Prove that $\tan(\frac{A}{2}) = \sqrt{\frac{1-\cos A}{1+\cos A}}$

(51) Find maximum and minimum value $5\sin x+12\cos x$.

(52) Find maximum and minimum values of $3\sin x+4\cos x-4$.

(53) Show that $\cos 4A-\cos 4B = 8(\cos A-\cos B)(\cos A+\cos B)(\cos A-\sin B)\times(\cos A+\sin B)$

(54) Show that $\cos^6 A-\sin^6 A = \cos 2A\left(1-\frac{1}{4}\sin^2 2A\right)$

(55) Show that $\sin^4\theta = \frac{3}{8}-\frac{1}{2}\cos 2\theta+\frac{1}{8}\cos 4\theta$

(56) Show that $\frac{1}{\tan 3A-\tan A}-\frac{1}{\cot 3A-\cot A} = \cot 2A$.

(57) Show that $\cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8} = \frac{3}{2}$

(58) Prove that $\tan 37\frac{1}{2}^\circ = \sqrt{6}+\sqrt{3}-\sqrt{2}-2$

(59) Prove that $\cos 10^\circ\cos 30^\circ\cos 50^\circ\cos 70^\circ = \frac{3}{16}$

(60) If an angle θ is divided into two parts α, β such that $\tan\alpha : \tan\beta = x : y$ then prove that $\sin(\alpha-\beta) = \frac{x-y}{x+y}\sin\theta$

(61) Prove that $\left(\frac{\cos A+\cos B}{\sin A-\sin B}\right)^n+\left(\frac{\sin A+\sin B}{\cos A-\cos B}\right)^n = 2\cot^n\left(\frac{A-B}{2}\right)$, if n is even

$= 0$ if n is odd

(62) Find $\dfrac{\cos 15 - \sin 15}{\cos 15 + \sin 15}$

(63) Find $\sin 105^\circ \cdot \cos 105^\circ$

(64) Find $2\sin 67\frac{1}{2} \cdot \cos 67\frac{1}{2}$

(65) If $A+B+C = \pi$ then prove that

(a) $\sin A + \sin B - \sin C = 4\sin\left(\frac{A}{2}\right) \cdot \sin\left(\frac{B}{2}\right) \cdot \cos\left(\frac{C}{2}\right)$

(b) $\sin 2A + \sin 2B + \sin 2C = 4\sin A \cdot \sin B \cdot \sin C$.

(c) $\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} = 4\sin\frac{\pi - A}{4} \cdot \sin\frac{\pi - B}{4} \cdot \sin\frac{\pi - C}{4} + 1$

(66) Find $\dfrac{1 - \tan 15^\circ}{1 + \tan 15^\circ}$

(67) If $\tan x + \tan y = 5$, $\tan x \cdot \tan y = \frac{1}{2}$ then find $\cot(x+y)$

(68) Prove that $2\sin 105^\circ \cdot \sin 15^\circ = \frac{1}{2}$.

(69) If $\cos A = \frac{1}{2}$, $\cos B = 1$ then prove that $\tan\frac{A+B}{2} \cdot \tan\frac{A-B}{2} = \frac{1}{3}$

(70) Find $\cos 15^\circ \cdot \cos 7\frac{1}{2}^\circ \cdot \sin 7\frac{1}{2}^\circ$

(71) Find $\sin 20^\circ (3 - 4\cos^2 70^\circ)$

(72) Prove that $\sqrt{3}(3\tan 10^\circ - \tan^3 10^\circ) = 1 - 3\tan^2 10^\circ$.

(73) Find $2\tan 7\frac{1}{2}^\circ \times \dfrac{1 - \tan^2(7\frac{1}{2})}{[1 + \tan^2(7\frac{1}{2})]^2}$

(74) If $\dfrac{1 + \sin A}{\cos A} = \sqrt{2} + 1$ then find $\dfrac{1 - \sin A}{\cos A}$.

(75) Find $\sin 35^\circ + \cos 55^\circ$

OBJECTIVES OF THE MONTH

(76) Find $\sin^2 24 - \sin^2 6$

(77) For what value of θ, $\cos 3\theta + \sin 3\theta$ is maximum

(78) Show that $\sin 10^\circ \cdot \sin 50^\circ \cdot \sin 60^\circ \cdot \sin 70^\circ = \frac{\sqrt{3}}{16}$.

(79) If $A+B+C=\pi$ then show that

(a) $\cos 2A+\cos 2B+\cos 2C+1+4\cos A.\cos B.\cos C=0$.

(b) $\cos^2 A+\cos^2 B+\cos^2 C+2\cos A.\cos B.\cos C=1$

(c) $\cos^2 A+\cos^2 B-\cos^2 C=1-2\sin A.\sin B.\sin C$

(d) $\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}=4\cos\frac{\pi-A}{4}.\cos\frac{\pi-B}{4}.\cos\frac{\pi-C}{4}$

(e) $\tan A+\tan B+\tan C=\tan A.\tan B.\tan C$.

(f) $\cos 2A-\cos 2B-\cos 2C=-1+4\cos A.\sin B.\sin C$.

(80) Prove that $\frac{1+\cos A}{\sin A}=\cot\left(\frac{A}{2}\right)$.

Hence deduce that $\cot 37\frac{1}{2}^\circ=\sqrt{6}-\sqrt{3}-\sqrt{2}+2$.

(81) Show that $2\sin\frac{\pi}{32}=\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}$

(82) If $\sin\theta+\operatorname{cosec}\theta=2$ then show that $\sin^n\theta+\operatorname{cosec}^n\theta=2$ for all +ve integers n.

(N) INVERSE TRIGONOMETRIC FUNCTION:

If $\boxed{\sin\theta=x \Leftrightarrow \sin^{-1}x=\theta}$. $\sin^{-1}x$ is called inverse circular function read as sine inverse x.

$\sin^{-1}x$ may be defined as an angle θ whose sine is x.

Similarly $\cos^{-1}x, \tan^{-1}x, \cot^{-1}x, \sec^{-1}x, \operatorname{cosec}^{-1}x$ are all ~~trigonometric f~~ Inverse trigonometric function

Note that $\boxed{\sin^{-1}x \neq (\sin x)^{-1}=\frac{1}{\sin x}}$

(O) Domain and range of T-function & I.T-functions.

functions	Domain	Range
$\sin\theta$	R	$[-1, 1]$
$\cos\theta$	R	$[-1, 1]$
$\tan\theta$	$R-\{(2n+1)\frac{\pi}{2}, n\in Z\}$	R
$\cot\theta$	$R-\{n\pi, n\in Z\}$	R
$\sec\theta$	$R-\{(2n+1)\frac{\pi}{2}, n\in Z\}$	$(-\infty,-1]\cup[1, \infty)$
$\operatorname{cosec}\theta$	$R-\{n\pi, n\in Z\}$	$(-\infty,-1]\cup[1,\infty)$

FUNCTION	domain	Range
$\sin^{-1}x$	$[-1,1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\cos^{-1}x$	$[-1,1]$	$[0, \pi]$
$\tan^{-1}x$	R	$(-\frac{\pi}{2}, \frac{\pi}{2})$
$\cot^{-1}x$	R	$(0, \pi)$.
$\sec^{-1}x$	$(-\infty, 1] \cup [1, \infty)$	$(0, \pi/2) \cup (\pi/2, \pi]$
$\text{cosec}^{-1}x$	$(-\infty, 1] \cup [1, \infty)$	$[-\pi/2, 0) \cup (0, \pi/2]$

PROPERTIES OF IT-functions:

P1: (a)
$\sin(\sin^{-1}x) = x$
$\cos(\cos^{-1}x) = x$
$\tan(\tan^{-1}x) = x$
$\cot(\cot^{-1}x) = x$
$\sec(\sec^{-1}x) = x$
$\text{cosec}(\text{cosec}^{-1}x) = x$

(b)
$\sin^{-1}(\sin x) = x$
$\cos^{-1}(\cos x) = x$
$\tan^{-1}(\tan x) = x$
$\cot^{-1}(\cot x) = x$
$\sec^{-1}(\sec x) = x$
$\text{cosec}^{-1}(\text{cosec}\, x) = x$.

P2: (a) $\sin^{-1}x = \text{cosec}^{-1}(\frac{1}{x})$
$\text{cosec}^{-1}x = \sin^{-1}(\frac{1}{x})$

(b) $\cos^{-1}x = \sec^{-1}(\frac{1}{x})$
$\sec^{-1}(x) = \cos^{-1}(\frac{1}{x})$

(c) $\tan^{-1}x = \cot^{-1}(\frac{1}{x})$, $\cot^{-1}(x) = \tan^{-1}(\frac{1}{x})$

P3: $\sin^{-1}(-x) = -\sin^{-1}x$
$\cos^{-1}(-x) = \pi - \cos^{-1}x$
$\tan^{-1}(-x) = -\tan^{-1}x$
$\cot^{-1}(-x) = -\cot^{-1}x$
$\sec^{-1}(-x) = \pi - \sec^{-1}x$
$\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x$.

P4: $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$
$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$
$\sec^{-1}x + \text{cosec}^{-1}x = \pi/2$.

P5: (a) $\sin^{-1}x = \cos^{-1}(\sqrt{1-x^2}) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

$= \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$

(b) $\cos^{-1}x = \sin^{-1}(\sqrt{1-x^2}) = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

$= \sec^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

(c) $\tan^{-1}x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \operatorname{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$

$= \sec^{-1}(\sqrt{1+x^2}) = \cot^{-1}\left(\frac{1}{x}\right)$

P6: (a) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, If $xy \le 1$

(b) $\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, If $xy > 1$

(c) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

P7: (a) $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

(b) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$

(c) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left[xy - \sqrt{(1-x^2)(1-y^2)}\right]$

(d) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left[xy + \sqrt{(1-x^2)(1-y^2)}\right]$

P8: (a) $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, $-1 \le x \le 1$

$= \tan^{-1}\left(\frac{2x}{1-x^2}\right)$; $-1 < x < 1$

$= \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, $(0 < x < \infty)$

(b) $2\sin^{-1}x = \sin^{-1}\left[2x\sqrt{1-x^2}\right]$

(c) $2\cos^{-1}x = \cos^{-1}\left[2x^2-1\right]$

(d) $3\sin^{-1}x = \sin^{-1}\left[3x-4x^3\right]$

(e) $3\cos^{-1}x = \cos^{-1}\left[4x^3-3x\right]$

(f) $3\tan^{-1}x = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

PROBLEMS

(83) Find the value of $\cos\tan^{-1}\cot\cdot\cos^{-1}\frac{\sqrt{3}}{2}$.

Soln $\cos\tan^{-1}\cot\cdot\cos^{-1}\frac{\sqrt{3}}{2}$

$= \cos\tan^{-1}\cot 30^\circ$

$= \cos\tan^{-1}\cdot\sqrt{3} = \cos(60^\circ) = \frac{1}{2}$ (Ans)

(84) Find $\sec^2(\tan^{-1}2) + \operatorname{cosec}^2(\cot^{-1}3)$

Soln $\sec^2(\tan^{-1}2) + \operatorname{cosec}^2(\cot^{-1}3)$

$= \sec^2\alpha + \operatorname{cosec}^2\beta$

Put $\tan^{-1}2 = \alpha$
$\Rightarrow \tan\alpha = 2$
& $\cot^{-1}3 = \beta$
$\Rightarrow \cot\beta = 3$

$= (1+\tan^2\alpha) + (1+\cot^2\beta)$

$= (1+2^2) + (1+3^2) = 1+4+1+9 = 15$.

(85) Find $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3$.

Soln $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3$

$= \pi + \tan^{-1}\left\{\frac{1+2}{1-1\times 2}\right\} + \tan^{-1}3$ $(\because 1\times 2 = 2 > 1)$

$= \pi + \tan^{-1}(-3) + \tan^{-1}3$

$= \pi - \tan^{-1}3 + \tan^{-1}3 = \pi$ (Ans)

(86) Solve $3\sin^{-1}\left(\frac{2x}{1+x^2}\right) - 4\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) + 2\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3}$

Soln $3\sin^{-1}\left(\frac{2x}{1+x^2}\right) - 4\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) + 2\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3}$

$\Rightarrow 3\times 2\tan^{-1}x - 4\times 2\tan^{-1}x + 2\times 2\tan^{-1}x = \frac{\pi}{3}$

$\Rightarrow 6\tan^{-1}x - 8\tan^{-1}x + 4\tan^{-1}x = \frac{\pi}{3}$

$\Rightarrow 2\tan^{-1}x = \frac{\pi}{3} \Rightarrow \tan^{-1}x = \frac{\pi}{6} \Rightarrow x = \tan\frac{\pi}{6}$

$\Rightarrow x = \tan 30^\circ = \frac{1}{\sqrt{3}}$ (Ans)

(87) If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$ then show that
$x^2+y^2+z^2+2xyz = 1$

Soln $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$

$\Rightarrow \cos^{-1}x+\cos^{-1}y=\pi-\cos^{-1}z$

$\Rightarrow \cos^{-1}\left[xy-\sqrt{(1-x^2)(1-y^2)}\right]=\cos^{-1}(-z)$

$\Rightarrow xy-\sqrt{(1-x^2)(1-y^2)}=-z \Rightarrow xy+z=\sqrt{(1-x^2)(1-y^2)}$

$\Rightarrow (xy+z)^2=(1-x^2)(1-y^2)$

$\Rightarrow x^2y^2+z^2+2xyz=1-y^2-x^2+x^2y^2$

$\Rightarrow z^2+2xyz=1-y^2-x^2 \Rightarrow x^2+y^2+z^2+2xyz=1$ □.

(88) Prove that $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{31}{17}$.

Soln LHS $=2\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{7}\right)$

$=\tan^{-1}\left\{\frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}\right\}+\tan^{-1}\left(\frac{1}{7}\right)=\tan^{-1}\left(\frac{4}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)$

$=\tan^{-1}\left\{\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\times\frac{1}{7}}\right\}$ $\left(\because \frac{4}{3}\times\frac{1}{7}=\frac{4}{21}<1\right)$

$=\tan^{-1}\left\{\frac{28+3}{21}\Big/\frac{21-4}{21}\right\}=\tan^{-1}\left\{\frac{31/21}{17/21}\right\}=\tan^{-1}\frac{31}{17}=$ RHS □.

(89) Find $\sin^{-1}\left(\frac{4}{5}\right)+2\tan^{-1}\left(\frac{1}{3}\right)$

Soln $\sin^{-1}\left(\frac{4}{5}\right)+2\tan^{-1}\left(\frac{1}{3}\right)=\sin^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left\{\frac{2\times\frac{1}{3}}{1+\left(\frac{1}{3}\right)^2}\right\}$

$=\sin^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{2/3}{10/9}\right)=\sin^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{3}{5}\right)$

$=\sin^{-1}\left[\frac{4}{5}\sqrt{1-\left(\frac{3}{5}\right)^2}+\frac{3}{5}\sqrt{1-(4/5)^2}\right]$

$=\sin^{-1}\left[\frac{4}{5}\times\frac{4}{5}+\frac{3}{5}\times\frac{3}{5}\right]=\sin^{-1}\left[\frac{16+9}{25}\right]=\sin^{-1}1$

$=90°=\frac{\pi}{2}$ (Ans)

(90) If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2}$ then show that $xy+yz+zx=1$.

Soln $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2}$.

$\Rightarrow \tan^{-1}x+\tan^{-1}y=\frac{\pi}{2}-\tan^{-1}z$

$\Rightarrow \tan^{-1}\left\{\frac{x+y}{1-xy}\right\}=\cot^{-1}z$

$\Rightarrow \tan^{-1}\left\{\frac{x+y}{1-xy}\right\}=\tan^{-1}\left(\frac{1}{z}\right)$

$\Rightarrow \frac{x+y}{1-xy}=\frac{1}{z}$

$\Rightarrow (x+y)z = 1-xy \Rightarrow xz+yz = 1-xy \Rightarrow xy+yz+zx = 1$ □.

(91) If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$ then prove that $x+y+z=xyz$

Soln $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$

$\Rightarrow \tan^{-1}x+\tan^{-1}y = \pi-\tan^{-1}z$

$\Rightarrow \tan^{-1}\left\{\dfrac{x+y}{1-xy}\right\} = \pi-\tan^{-1}z$

$\Rightarrow \dfrac{x+y}{1-xy} = \tan\{\pi-\tan^{-1}z\}$

$\Rightarrow \dfrac{x+y}{1-xy} = -\tan(\tan^{-1}z) \Rightarrow \dfrac{x+y}{1-xy} = -z$

$\Rightarrow x+y = -z(1-xy) \Rightarrow x+y = -z+xyz$

$\Rightarrow x+y+z = xyz$ □.

HOME TASK

(92) Prove that $2\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{4} = \tan^{-1}\left(\frac{8}{53}\right)$

(93) Prove that $2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7} = \frac{\pi}{4}$

(94) Find $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}$.

(95) prove that

$\sin^2(\sin^{-1}x+\sin^{-1}y+\sin^{-1}z) = \cos^2(\cos^{-1}x+\cos^{-1}y+\cos^{-1}z)$

(96) prove that

$\tan(\tan^{-1}x+\tan^{-1}y+\tan^{-1}z) = \cot(\cot^{-1}x+\cot^{-1}y+\cot^{-1}z)$

(97) If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\pi$ then show that

$x\sqrt{1-x^2}+y\sqrt{1-y^2}+z\sqrt{1-z^2} = 2xyz$.

(98) In any triangle ABC if $\angle A = 90°$ then prove that $\tan^{-1}\left(\frac{b}{a+c}\right)+\tan^{-1}\frac{c}{a+b} = \frac{\pi}{4}$, a, b, c are sides

(99) Prove that

$2\tan^{-1}(\operatorname{cosec}\tan^{-1}x - \tan\cot^{-1}x) = \tan^{-1}x$.

(100) solve $\sin^{-1}x + \sin^{-1}(1-x) = \pi/2$

(101) solve $\sin^{-1}\left(\frac{2a}{1+a^2}\right) + \sin^{-1}\left(\frac{2b}{1+b^2}\right) = 2\tan^{-1}x$

(102) $\cos^{-1}x + \sin^{-1}\left(\frac{x}{2}\right) = \frac{\pi}{6}$

(103) find $\cos\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$

(104) Find $\sin^{-1}\cos x - \cos^{-1}\sin 2x$

(105) Find $\sin^{-1}1 - \tan^{-1}1$.

(106) Find $\tan^{-1}\sqrt{3} + \tan^{-1}\frac{1}{\sqrt{3}}$

(107) If $A = \tan^{-1}x$ then find $\sin 2A$.

(108) If $\sin^{-1}x = \frac{\pi}{5}$, $x \in (-1, 1)$ then find $\cos^{-1}x$

(109) Find $\tan^{-1}\left(2\cos\frac{\pi}{3}\right)$

(110) If $x+y=4$, $xy=1$ then find $\tan^{-1}x + \tan^{-1}y$.

(111) find principal values of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

(112) If $\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$ then find x.

(113) Find the value of $\sin(\tan^{-1}x + \cot^{-1}x)$