COMPLEX NUMBER:

MATH-III/I

A: REAL AND IMAGINARY NUMBERS:

i) Real number is the union of rationals and irrationals. The set of all real numbers is denoted by R.

$\boxed{R = Q \cup Q'}$ where Q = set of all rationals. and Q' = set of all irrationals.

(ii) Note that if $\boxed{a \in R \text{ then } a^2 \geq 0}$

(iii) IMAGINARY NUMBERS:

(a) The square root of any negative real number is called an imaginary number. In General $\boxed{\sqrt{-a},\ a \in R,\ a > 0}$ is an imaginary Number.

(b) $\sqrt{-1}, \sqrt{-2}, \sqrt{-3}, \sqrt{-4}, \sqrt{-\frac{3}{2}}, \sqrt{-\sqrt{2}}$ ---- are imaginary numbers.

(c) $\sqrt{-1}$ is called the unit imaginary number or basic imaginary Number for every imaginary Number can be expressed in terms of $\sqrt{-1}$. It is denoted by i.

$\therefore$ $\boxed{i = \sqrt{-1}}$ & $\boxed{i^2 = -1}$

(d) Note that $\boxed{\sqrt{-a} = i\sqrt{a}}$

(e) $\boxed{i^{4n} = 1,\ i^{4n+k} = i^k}$

(f) $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ This formula is not true only when both a and b are (-ve).

If both a and b are -ve then simplify $\sqrt{a} \times \sqrt{b}$ individually.

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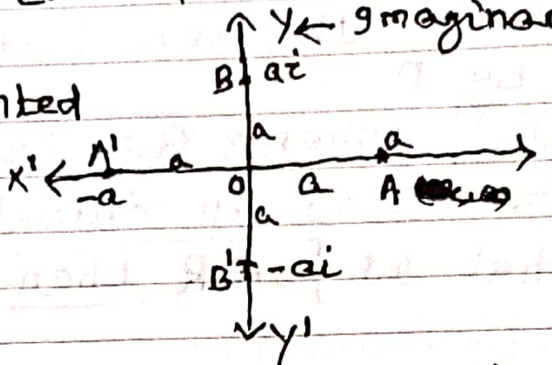
MATH-111/2

(iv) Geometrical representation of imaginary Numbers:

(a) Let's consider two co-ordinate axes, X'OX and YOY'.

Let 'a' is a real number represented by the point A on x-axis. Then $OA = a$.



If we rotate $\overline{OA}$ through an angle of 180° in anti clock wise direction, then it coincides with the segment $\overline{OA'}$ where $OA' = a$ but point A' is $-a$

$(-a) = i \times i\, a$

Thus if we rotate $\overline{OA}$ through 180° angle a is multiplied by i twice.

If we rotate $\overline{OA}$ through 90° angle a is multiplied with i and we get ai on y axis at point B. If we multiply a thrice by i then we get $-ai$ on y axis at B'.

(b) Thus every imaginary numbers are represented by y-axis and real numbers are on x-axis. So x-axis is called real axis and y-axis is called imaginary axis.

Problems:

(1) Find $i^{17} + i^{20} - i^{13}$.

(2) Find $(-i)^{4n+3}$. (3) Find $(i)^{4n+3}$.

(4) Find i^{5000}, i^{306}, i^{1420}, i^{1421}, i^{1422}, i^{1423}.

(5) Multiply $\sqrt{-2}$, $\sqrt{-3}$

(6) Multiply $\sqrt{-4} \times \sqrt{-5}$ (7) Find $i^{215} + i^{218}$.

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(B) COMPLEX NUMBER. (cartesian form)

(i) The number of the form $x+iy$, $x,y \in R$, $i=\sqrt{-1}$ is called complex number. If z is a complex number then we write it as $\boxed{z = x+iy = (x,y)}$.

Set of all complex number is denoted by C.

$\therefore C = \{z \mid z = x+iy,\ x,y \in R,\ i=\sqrt{-1}\}$. This form is called cartesian form.

Ex: $2+3i$, $-7i+3$, $\frac{1}{2}+\frac{3}{4}i$, 0, 5, $\frac{1}{2}$, $-3i$, $-\frac{1}{2}i$ etc.

(ii) If $z = x+iy$ is a complex number then

Real part of $z = Re.(z) = x$

Imaginary part of $z = Im.(z) = y$.

(iii) The complex number whose real part is zero is called purely imaginary and whose imaginary part is zero is called purely real.

Ex: purely imaginary $\rightarrow 0, 5i, 3i, -\frac{1}{2}i$ ......

purely real $\rightarrow 0, 3, -4, \frac{1}{2}$ ....

(iv) Modulus of a complex number.

$\rightarrow$ If $z = x+iy$ is a complex number the modulus of z is denoted by $|z|$ and defined by $\boxed{|z| = \sqrt{x^2+y^2}}$

Properties: ~~(a)~~ (a) $|z_1 z_2| = |z_1|\cdot|z_2|$ ~~(c)~~

(b) $|z_1+z_2| \le |z_1|+|z_2|$ (c) $|z_1-z_2| \le |z_1|+|z_2|$

(d) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

(v) CONJUGATE OF A COMPLEX NUMBER:

If $z = x+iy$ is a complex number then conjugate of z is denoted by $\bar{z}$ and $\boxed{\bar{z} = x-iy}$

(vi) Reciprocal of a C.N. If z is a complex number then then it $\boxed{\text{reciprocal is } \frac{1}{z}}$, $z \neq 0$

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(v) Equality of two C.N:

Two complex number $a+bi$ and $c+di$ are equal iff $a=c$ and $b=d$. i.e

$$\boxed{a+bi = c+di \Leftrightarrow a=c \text{ and } b=d.}$$

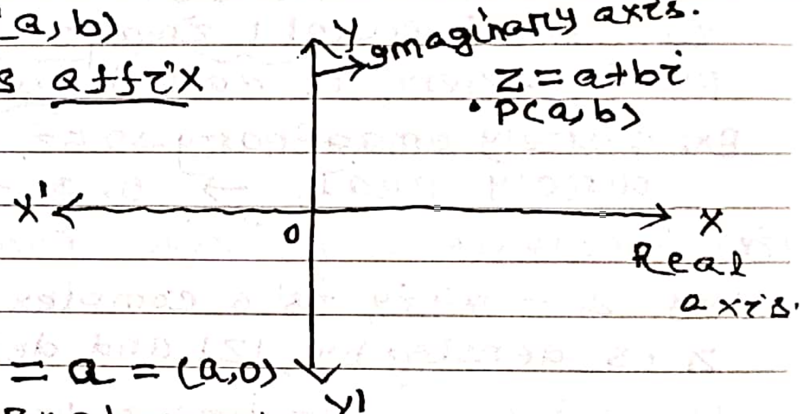
(vi) Geometrical representation of complex Number.

→ To every complex Number $z = a+ib$, there is associated a unique order pair of real number (a,b). Thus we can write

$$\boxed{z = a+bi = (a,b)}$$

→ Let's consider two co-ordinate axes XOX' and YOY'. Then any number $z = a+bi$ can be represented by unique point $P(a,b)$

$z = a+bi$ is known as affix of the point (a,b).



→ If $b=0$ then $z = a = (a,0)$ which is a purely real number. But $P(a,0)$ is a point on x-axis. so purely real numbers are represented on x-axis. Hence x-axis is called real axis.

→ If $a=0$ then $z = bi = (0,b)$ which is a purely imaginary number. But $P(0,b)$ is a point on y-axis. so purely imaginary Numbers are represented on y-axis. Thus y-axis is called imaginary axis.

→ Thus there is a one-to-one correspondence between the set of all points in plane and set of all complex Numbers.

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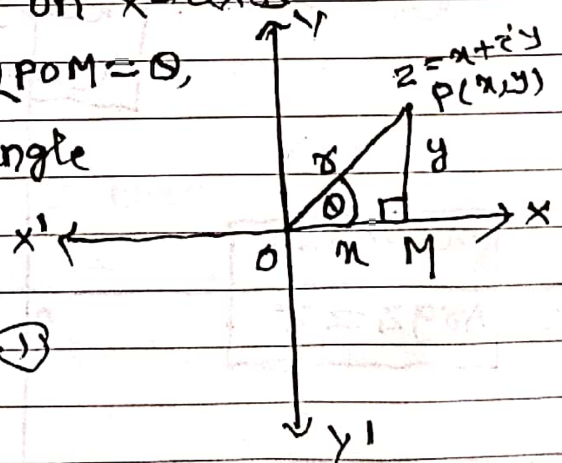
→ The representation of complex numbers as points in plane forms an Argand diagram. The plane on which complex numbers are represented is called complex or Argand's or Gaussian plane.

(vi) Argument/Amplitude of a C.N (polar form)

(a) Argument of a complex number z is the angle between the positive real axis and the line joining origin with z measured in anticlockwise direction. It is denoted by $\operatorname{Arg}(z)$ or $\operatorname{amp}(z)$.

(b) Let $\arg z = \theta$. Then the unique values of θ for which $-\pi < \theta \le \pi$ is called principal values of argument. The general value of argument is $2n\pi+\theta$ where θ = principal value, n = integer.

(c) Let $z = x+iy$ is a complex number represented by point P. Draw $\perp$ar PM on x-axis. Join OP. ~~Then~~ Let $m\angle POM = \theta$, $OP = r$.



Then in the right angle Triangle OMP

$$\cos\theta = \frac{OM}{OP} = \frac{x}{r}$$

$$\Rightarrow x = r\cos\theta \quad \text{--- (1)}$$

$$\sin\theta = \frac{PM}{OP} = \frac{y}{r}$$

$$\Rightarrow y = r\sin\theta \quad \text{--- (2)}$$

Now $x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\sin^2\theta+\cos^2\theta)$

$$\Rightarrow x^2+y^2 = r^2 \Rightarrow r = \sqrt{x^2+y^2} = |z| \quad \text{--- (3)}$$

Now $\dfrac{y}{x} = \dfrac{r\sin\theta}{r\cos\theta} = \tan\theta$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\boxed{\operatorname{Arg} z = \theta = \tan^{-1}\frac{y}{x}} \quad \text{--- (4)}$$

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(d) Now $z = x + iy$

$\Rightarrow z = r\cos\theta + i \cdot r\sin\theta$

$\Rightarrow \boxed{z = r(\cos\theta + i\sin\theta)}$ — (5)

where $r = |z|$, $\theta = Arg(z)$ (principal Argument)

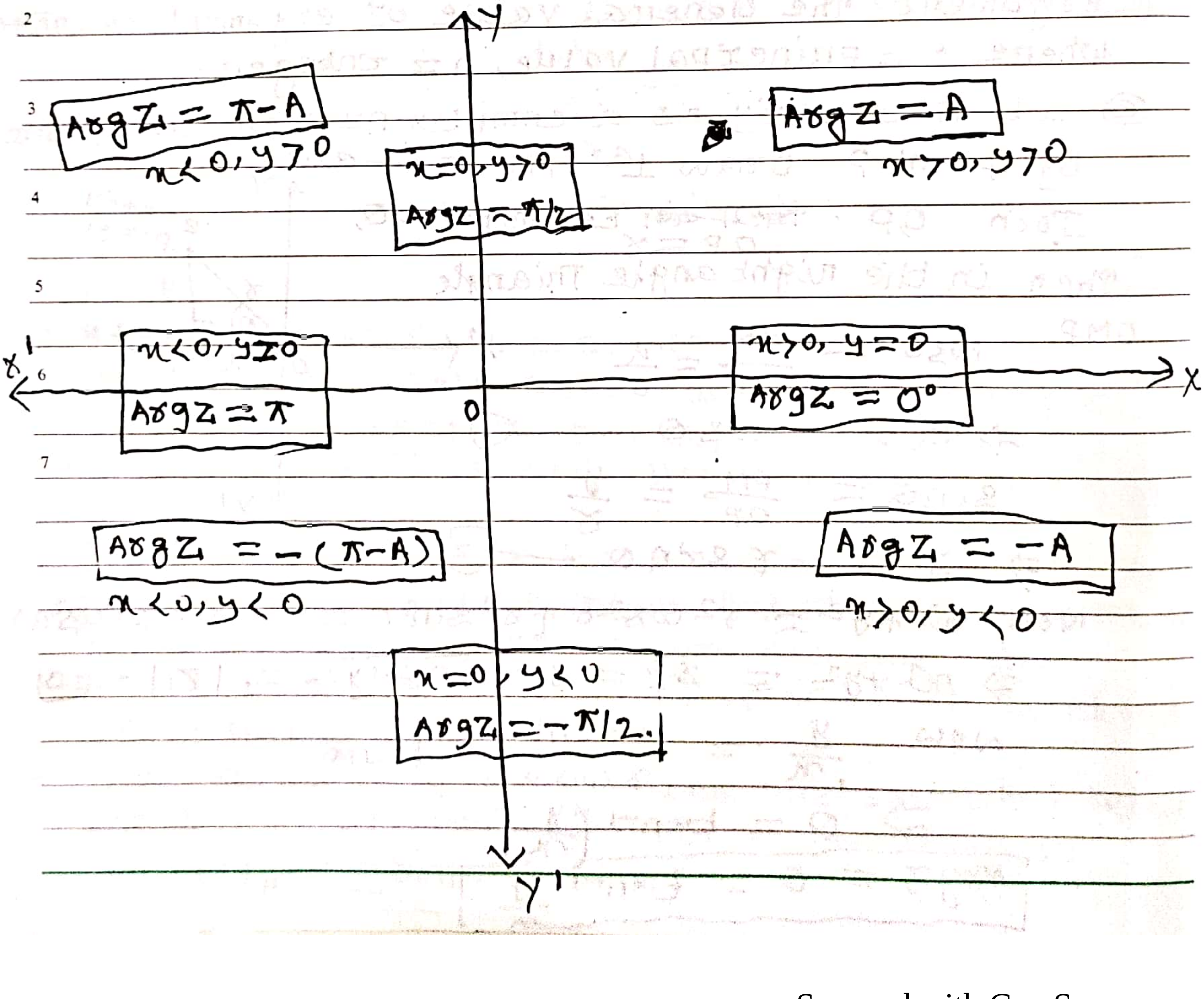
Eqn (5) is called polar form.

(e) Procedure to find Arg(z)

→ Let $z = x + iy$ be given complex number.

→ Find $A = \tan^{-1}\left|\frac{y}{x}\right|$

→ Then follow following fig. to find arg z.



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NOTE (i) $\cos\theta + i\sin\theta = e^{i\theta}$ (Eulers formula)

(ii) $z = re^{i\theta}$ (Exponential form). (iii) $\mathrm{Arg}(0)$ is ∞.

(C) PROPERTIES OF COMPLEX Number: If z, z_1, z_2 are complex numb

(a) (i) $|\bar{z}| = |z|$ (ii) $\overline{(\bar{z})} = z$ (iii) $(z+\bar{z}) = 2\,\mathrm{Re}(z)$ (iv) $(z-\bar{z}) = 2i\,\mathrm{Im}(z)$

(v) $(z\cdot\bar{z}) = |z|^2$ (vi) $\mathrm{Re}(z) \le |z|$, $\mathrm{Im}(z) \le |z|$ (vii) $\overline{(z_1+z_2)} = \bar{z}_1 + \bar{z}_2$

(viii) $\overline{(z_1-z_2)} = \bar{z}_1 - \bar{z}_2$ (ix) $\overline{z_1\cdot z_2} = \bar{z}_1\cdot\bar{z}_2$ (x) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$.

(xi) $|z_1+z_2| \le |z_1| + |z_2|$ (xii) $|z_1-z_2| \le |z_1| + |z_2|$

(xiii) $|z_1\cdot z_2| = |z_1|\cdot|z_2|$ (xiv) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

(xv) $\mathrm{Arg}(z_1\cdot z_2) = \mathrm{Arg}\,z_1 + \mathrm{Arg}(z_2)$

(xvi) $\mathrm{Arg}\left|\frac{z_1}{z_2}\right| = \mathrm{Arg}\,z_1 - \mathrm{Arg}(z_2)$

(b) (i) closure property: If $z_1, z_2 \in \mathbb{C} \Rightarrow z_1+z_2,\ z_1-z_2,\ z_1\cdot z_2,\ z_1/z_2 \in \mathbb{C}$.

(ii) commutative property If $z_1, z_2 \in \mathbb{C}$ then $z_1+z_2 = z_2+z_1$ and $z_1\cdot z_2 = z_2\cdot z_1$

(iii) Associative law: If $z_1, z_2, z_3 \in \mathbb{C}$ then

$(z_1+z_2)+z_3 = z_1+(z_2+z_3)$ & $(z_1\cdot z_2)\cdot z_3 = z_1\cdot(z_2\cdot z_3)$

(iv) Identity Law: If $z \in \mathbb{C}$ then there exist $0, 1 \in \mathbb{C}$ s.t $z+0 = 0+z = z$

& $z\cdot 1 = 1\cdot z = z$.

(v) Inverse Law: If $z \in \mathbb{C}$ then $\exists\ -z \in \mathbb{C}$ s.t

$$z + (-z) = (-z) + z = 0$$

If $z \neq 0 \in \mathbb{C}$ then $\exists\ z^{-1} \in \mathbb{C}$ s.t

$$z\cdot z^{-1} = z^{-1}\cdot z = 1$$

(vi) Distributive Law: $z_1(z_2+z_3) = z_1\cdot z_2 + z_1\cdot z_3$

$(z_1+z_2)z_3 = z_1 z_3 + z_2 z_3$.

MATH-III/8

PROBLEMS:

(7) Find a and b if $\frac{5+3i}{5-3i} = a+bi$

(8) Convert $\frac{2+3i}{2-3i}$ into $x+iy$ form and hence find its real and imaginary part.

(9) Find real and imaginary part of $(7+3i)(4-5i)$

(10) Convert $\frac{7-3i}{2+5i}$ into $x+iy$ form.

(11) Convert $\frac{(2+7i)(1-i)}{(3-5i)(2+i)}$ into $x+iy$ form.

(12) Find a and b if (i) $\frac{2+3i}{5-7i} = a+bi$

(ii) $\frac{4-3i}{4+3i} = a+bi$

(13) Find x, y if

$(x-2y)+3yi = 4-6i$

(14) Find Modulus of (a) $(1+i)(1+2i)(1+3i)$

(b) $\frac{5+12i}{3+4i}$

(15) Find conjugate of (a) $\frac{4+3i}{5-7i}$

(b) $(2+3i)(7-5i)$

(16) Find reciprocal or multiplicative inverse of

(a) $\frac{7+2i}{3-4i}$ (b) $(2+3i)$ (c) i

(17) Find Argument of (a) $\left(\frac{-1+i\sqrt{3}}{2}\right)$ (b) $1+i\sqrt{3}$

(c) $(1-i\sqrt{3})$ (d) 2 (e) (-3) (f) i (g) $-5i$

MATH-iii/9:

(18) convert into polar form:
(a) $1+i$ (b) $-1+i\sqrt{3}$ (c) $1+i\sqrt{3}$ (d) $1-i\sqrt{3}$

(19) Find x, y if $(x+y)+i(x-y)=12+6i$

(C) SQUARE Root of a Complex Numbers

Square root of a complex number is a complex number. To find square root of a complex number apply any one of following methods.

Method-i (General method) In this method to find $\sqrt{a+bi}$, take $\sqrt{a+bi}=x+iy$ and to find $\sqrt{a-bi}$, take $\sqrt{a-bi}=x-iy$. Then proceed to find x and y.

Method-ii (Shortcut Method / Perfect square method) To find $\sqrt{a+bi}$, splits 'a' as difference of two numbers x & y where $x=\frac{r+a}{2}$ & $y=\frac{r-a}{2}$, $r=\sqrt{a^2+b^2}$

Method-iii use Demovir's theorem.

(20) Find square roots of (a) $-5+12\sqrt{-1}$ (b) $3+4i$ (c) $3-4i$.

(D) Cube roots of unity

(21) Find cube roots of unity.

(22) Find cube roots of 8.

Properties: (i) $1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$ are three cube roots of unity.

(ii) If $\omega=\frac{-1+i\sqrt{3}}{2}$ then $\omega^2=\frac{-1-i\sqrt{3}}{2}$,

$\therefore 1, \omega, \omega^2$ are cube roots of unity.

MATH-III/10:

(iii) $\omega^3 = 1$, $\omega^{3n} = 1$, $\omega^{3n+k} = \omega^k$.

(iv) $1+\omega+\omega^2 = 0$

(22) Prove that

(a) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5) = 9$

(b) $(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)$ ----- to $2n$ factors $= 2^{2n}$.

(c) $(1-\omega+\omega^2)^5 + (1+\omega-\omega^2)^5 = 32$

(d) $(2+5\omega+2\omega^2)^6 = (2+2\omega+5\omega^2)^6 = 729$.

(e) Find $(1+\omega)^5$.

(f) $(1-\omega+\omega^2)^7 + (1+\omega-\omega^2)^7 = 128$

(g) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8) = 9$

(h) $(2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11}) = 49$.

(E) DEMOIVRE'S THEOREM.

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

NOTE: (i) $(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$

(ii) $(\cos\theta - i\sin\theta)^{n} = \cos n\theta - i\sin n\theta$

(iii) $(\cos\theta - i\sin\theta)^{-n} = \cos n\theta + i\sin n\theta$

APPLICATION: It can be applied to find nth. root of a complex number. To find nth. root of a complex number, convert it into polar form.

Let $z = x+iy = r(\cos\theta + i\sin\theta)$ be the polar form where $r = \sqrt{x^2+y^2}$, $\theta = \text{Arg}\, z$.

Then $z = r\{\cos(2k\pi+\theta) + i\sin(2k\pi+\theta)\}$

$$\Rightarrow \boxed{z^{\frac{1}{n}} = r^{\frac{1}{n}}\left\{\cos\frac{2k\pi+\theta}{n} + i\sin\frac{2k\pi+\theta}{n}\right\}}$$

where $k = 0, 1, 2, 3, \ldots, n-1$

PRODUCT RULE

$$(\cos\alpha + i\sin\alpha)\cdot(\cos\beta + i\sin\beta) = \cos(\alpha+\beta) + i\sin(\alpha+\beta).$$

This formula is applicable for two or more than two factors.

(24) Solve $z^3 = 1+i$ (25) Solve $z^7 = 1$ (26) Find 4th root of $(1-i)$

(27) Solve $x^9 + x^5 - x^4 = 1$.

(28) Find the value of $\left(\sin\frac{\pi}{3} + i\cos\frac{\pi}{3}\right)^3$ using Demoivre's theorem.

(29) Find $\sqrt{3+4i}$ using De movire's thm.

(30) If $x + \frac{1}{x} = 2\cos\theta$ then prove that $x^n + \frac{1}{x^n} = 2\cos n\theta$;

(31) Show that $\left(\dfrac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n = \cos\left(\frac{n\pi}{2}-n\theta\right) + i\sin\left(\frac{n\pi}{2}-n\theta\right)$

(32) If α, β are roots of $x^2 - 2x + 4 = 0$ then show that $\alpha^n + \beta^n = 2^{n+1}\cos\left(\frac{n\pi}{3}\right)$.

(33) If $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$ then show that (a) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$

$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

(b) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 3/2$.

(34) For positive integer n, show that

(a) $(1+i)^n + (1-i)^n = 2^{\frac{n+2}{2}}\cos\left(\frac{n\pi}{4}\right)$

(b) $(1+\sqrt{3}i)^n + (1-\sqrt{3}i)^n = 2^{n+1}\cos\left(\frac{n\pi}{3}\right)$

(35) Find $\mathrm{Arg}(\omega) + \mathrm{Arg}(\omega^2)$.

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MATH-111/12

SOLVED PROBLEMS

① $i^{17} + i^{20} - i^{13}$

$= i^{4\times4+1} + i^{4\times5} - i^{4\times3+1}$

$= i^1 + 1 - i^1 = i + 1 - i = 1$ (Ans)

② $(-i)^{4n+3} = (-i)^{4n} \times (-i)^3$

$= \{(-i)^2\}^{2n} \times (-1)^3 \times i^3$

$= (i^2)^{2n} \times (-1) \times i^2 \times i$

$= (-1)^{2n} \times (-1) \times (-1) \times i$

$= (1) \times (1) \times i = i$ (Ans)

⑤ $\sqrt{-2} \times \sqrt{-3} = i\sqrt{2} \times i\sqrt{3}$

$= (i \times i) \times (\sqrt{2} \times \sqrt{3})$

$= i^2 \times \sqrt{2\times3} = (-1)\sqrt{6} = -\sqrt{6}$ (Ans)

⑦ $\frac{5+3i}{5-3i} = a+bi \Rightarrow \frac{(5+3i)(5+3i)}{(5-3i)(5+3i)} = a+bi$

$\Rightarrow \frac{(5+3i)^2}{(5)^2-(3i)^2} = (a+bi)$

$\Rightarrow \frac{25+9i^2+30i}{25-9i^2} = a+bi$

$\Rightarrow \frac{25+9(-1)+30i}{25-9(-1)} = a+bi$

$\Rightarrow \frac{25-9+30i}{25+9} = a+bi$

$\Rightarrow \frac{16+30i}{34} = a+bi$

$\Rightarrow \frac{16}{34} + \frac{30}{34}i = a+bi$

MATH-111/13

$$\frac{8}{17} + \frac{15}{17}i = a + bi$$

$$\Rightarrow a = \frac{8}{17},\ bi = \frac{15}{17}i$$

$$\Rightarrow a = \frac{8}{17},\ b = \frac{15}{17} \text{ (Ans)}$$

~~14(b) Let~~ $\cancel{z = \frac{5+12i}{5-7i}}$

$\cancel{= (5+12i)(5+7i)}$

14(b) Let $z = \dfrac{5+12i}{3+4i}$

$$= \frac{(5+12i)(3-4i)}{(3+4i)(3-4i)}$$

$$= \frac{15-20i+36i-48i^2}{(3)^2-(4i)^2}$$

$$= \frac{15+16i-48(-1)}{9-16(-1)} = \frac{63+16i}{25}$$

$$= \frac{63}{25} + \frac{16}{25}i$$

Here $x = \frac{63}{25},\ y = \frac{16}{25}$

$$|z| = \sqrt{x^2+y^2} = \sqrt{\left(\frac{63}{25}\right)^2 + \left(\frac{16}{25}\right)^2}$$

$$= \sqrt{\frac{63^2+16^2}{25^2}} = \sqrt{\frac{3969+256}{25^2}} = \sqrt{\frac{4225}{25^2}}$$

$$= \frac{65}{25} = \frac{13}{5} \text{ (Ans)}$$

15(a) Let $z = \frac{4+3i}{5-7i}$

9

$= \frac{(4+3i)(5+7i)}{(5-7i)(5+7i)}$

10

$= \frac{20+28i+15i+21i^2}{(5)^2-(7i)^2}$

11

12

$= \frac{20+43i+21(-1)}{25-49(-1)}$

1

$= \frac{-1+43i}{74} = -\frac{1}{74}+\frac{43}{74}i$

2

conjugate of z is $\bar{z} = -\frac{1}{74}-\frac{43i}{74}$

3

$\therefore$ conjugate of $\frac{4+3i}{5-7i}$ is $-\frac{1}{74}-\frac{43i}{74}$ (Ans)

4

16(c) Let $z = i$

5

Reciprocal of $z = \frac{1}{z} = \frac{1}{i}$

6

$= \frac{1\times i}{i\times i} = \frac{i}{i^2} = \frac{i}{-1} = -i$ (Ans)

7

17(a) Let $z = \frac{-1+i\sqrt{3}}{2} = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$

Here $x = -\frac{1}{2}$, $y = \frac{\sqrt{3}}{2}$

$\therefore (x,y) \in Q_2$

MATH-111/15

$A = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{\sqrt{3}/2}{-1/2}\right| = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

$\because (x,y) \in Q_2$

$\text{Arg} z.\ \theta = \pi - A = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ (Ans)

18(a) Let $z = 1+i$

Here $x=1, y=1$

$\therefore r = |z| = \sqrt{x^2+y^2} = \sqrt{(1)^2+(1)^2} = \sqrt{2}$

$(x,y) \in Q_1$.

$A = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{1}{1}\right| = \tan^{-1}1 = \frac{\pi}{4}$.

$\because (x,y) \in Q_1$

$\theta = A = \frac{\pi}{4}$.

Now polar form of z is

$z = r(\cos\theta + i\sin\theta)$

$\Rightarrow 1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$.

20 (b) Method-1

Let $\sqrt{3+4i} = x+iy$ ——— (1)

$\Rightarrow 3+4i = x^2+i^2y^2+2xyi$

$\Rightarrow 3+4i = (x^2-y^2)+(2xy)i$

$\Rightarrow x^2-y^2 = 3$ ——— (2)

& $2xy = 4$ ——— (3)

MATH-111/16

Now $x^2+y^2 = \sqrt{(x^2-y^2)^2+4x^2y^2}$

$= \sqrt{(x^2-y^2)^2+(2xy)^2}$

$= \sqrt{(3)^2+(4)^2}$

$= \sqrt{9+16} = \sqrt{25} = \pm 5.$

$\therefore x^2+y^2 = 5$ —— (4) $\quad (\because x^2+y^2 \geq 0)$

Adding (2) & (4),

$2x^2 = 8 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2.$

When $x = 2,$

$y = \frac{4}{2x}$ From (3)

$= \frac{4}{2\times 2} = 1$

$\therefore \sqrt{3+4i} = x+iy = 2+i$ —— (5)

when $x = -2$

$y = \frac{4}{2x} = \frac{4}{2(-2)} = -1$

$\therefore \sqrt{3+4i} = x+iy = -2-i$ —— (6)

From (5) & (6)

$\sqrt{3+4i} = \pm 2 \pm i = \pm(2+i)$ (Ans)

MATH-III/17.

Method-2

$\sqrt{3+4i}$

$\sqrt{4-1+4i}$

$=\sqrt{(2)^2+(i)^2+2(2)(i)}$

$=\sqrt{(2+i)^2} = \pm(2+i)$ (Ans)

Rough.

$\because r=\sqrt{3^2+4^2}$
$=5$

$\frac{r+a}{2}=\frac{5+3}{2}$
$=4$

$\frac{r-a}{2}=\frac{5-3}{2}$
$=1$

(21) Let cube root of unity be x

Then $\sqrt[3]{1}=x$

$\Rightarrow x^3=1 \Rightarrow x^3-1=0$

$\Rightarrow (x-1)(x^2+x+1)=0 \Rightarrow x-1=0$ or $x^2+x+1=0$

$\Rightarrow x=1$ or $x=\frac{-1\pm\sqrt{(1)^2-4.1.1}}{2\times1}$

$\Rightarrow x=1$ or $x=\frac{-1\pm\sqrt{-3}}{2}$

$\Rightarrow x=1$ or $x=\frac{-1\pm i\sqrt{3}}{2}$

$\Rightarrow x=1$ or $x=\frac{-1+i\sqrt{3}}{2}$ or $x=\frac{-1-i\sqrt{3}}{2}$

$\therefore$ cube roots of unity are $1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$ (Ans)

(23) (a) $(2+5\omega+2\omega^2)^6 = (2+2\omega^2+5\omega)^6$

$=\{2(1+\omega^2)+5\omega\}^6 = \{2(-\omega)+5\omega\}^6$ $(\because 1+\omega+\omega^2=0)$

$=(3\omega)^6 = 729\times\omega^6 = 729\times\omega^{3\times2}$

$=729\times1=729$ —— (1) $(\because \omega^{3n}=1)$

MATH-111/18

Again $(2+2\omega+5\omega^2)^6$

$= \{2(1+\omega)+5\omega^2\}^6$

$= \{2(-\omega^2)+5\omega^2\}^6$ $(\because 1+\omega+\omega^2=0)$

$= \{3\omega^2\}^6 = 729\times\omega^{12}$

$= 729\times\omega^{3\times4} = 729\times1$ $(\because \omega^{3n}=1)$

$= 729$ —— (2)

From (1) & (2) we have

$(2+5\omega+2\omega^2)^6 = (2+2\omega+5\omega^2)^6 = 729$ $\square$

(24) $z^3 = 1+i$ —— (1)

$\Rightarrow z = \sqrt[3]{1+i} = (1+i)^{\frac{1}{3}} = z_1^{\frac{1}{3}}$ (say) —— (1)

where $z_1 = 1+i$

Here $x=1, y=1 \Rightarrow r = \sqrt{x^2+y^2} = \sqrt{1^2+1^2} = \sqrt{2}$

$A = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{1}{1}\right| = \tan^{-1}1 = \frac{\pi}{4}$

$\because (x,y)\in Q_1$, $Arg(z_1)$

$\theta = A = \frac{\pi}{4}$

Now using ~~Demoivre~~ Demoivre's thm

$z = z_1^{\frac{1}{3}} = r^{\frac{1}{3}}\left\{\cos\left(\frac{2k\pi+\theta}{3}\right)+i\sin\left(\frac{2k\pi+\theta}{3}\right)\right\}$

$\Rightarrow z = (\sqrt{2})^{\frac{1}{3}}\left\{\cos\left(\frac{2k\pi+\frac{\pi}{4}}{3}\right)+i\sin\left(\frac{2k\pi+\pi/4}{3}\right)\right\}$

$\Rightarrow \cancel{z_1}\ 2^{\frac{1}{6}}\{\cancel{(8k+1)\frac{\pi}{12}}$

MATH-iii/19

$\Rightarrow z = 2^{\frac{1}{6}}\left\{\cos\frac{(8k+1)\pi}{12} + i\sin\frac{(8k+1)\pi}{12}\right\}$

where $k = 0, 1, 2.$ (Ans)

(28) $\left(\sin\frac{\pi}{3} + i\cos\frac{\pi}{3}\right)^3$

$= \left\{\cos\left(\frac{\pi}{2}-\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{2}-\frac{\pi}{3}\right)\right\}^3$

$= \left\{\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right\}^3$

$= \left\{\cos\left(3\times\frac{\pi}{6}\right) + i\sin\left(3\times\frac{\pi}{6}\right)\right\}$ (using Demoivre's thm)

$= \left\{\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right\} = 0 + i\times 1 = i$ (Ans).

(30) $x + \frac{1}{x} = 2\cos\theta$

$\Rightarrow \frac{x^2+1}{x} = 2\cos\theta$

$\Rightarrow x^2+1 = 2x\cos\theta$

$\Rightarrow x^2 - 2x\cos\theta + 1 = 0$

$\Rightarrow x = \frac{-(-2\cos\theta) \pm \sqrt{(-2\cos\theta)^2 - 4.1.1}}{2\times 1}$

$= \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$

$= \frac{2\cos\theta \pm 2\sqrt{\cos^2\theta - 1}}{2}$

$= \cos\theta \pm \sqrt{-\sin^2\theta}$

MATH-iii/20

12 December 2016							
Wk	M	T	W	T	F	S	S
49				1	2	3	4
50	5	6	7	8	9	10	11
51	12	13	14	15	16	17	18
52	19	20	21	22	23	24	25
53	26	27	28	29	30	31	

01 January 2017							
Wk	M	T	W	T	F	S	S
53	30	31					1
1	2	3	4	5	6	7	8
2	9	10	11	12	13	14	15
3	16	17	18	19	20	21	22
4	23	24	25	26	27	28	29

$= \cos\theta \pm i\sin\theta$

9 If $x = \cos\theta + i\sin\theta$ then

10 $x^n + \frac{1}{x^n} = (\cos\theta + i\sin\theta)^n + \frac{1}{(\cos\theta + i\sin\theta)^n}$

11 $= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$

12 $= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

(using De-moivre's thm)

1 $= 2\cos n\theta$

2 Similarly taking $x = \cos\theta - i\sin\theta$ we

3 can prove $x^n + \frac{1}{x^n} = 2\cos n\theta$ □

(31) L.H.S $= \cancel{\cos\theta +} \left(\frac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta}\right)^n$ 4

5 $= \left[\frac{1+\cos(\frac{\pi}{2}-\theta) + i\sin(\frac{\pi}{2}-\theta)}{1+\cos(\frac{\pi}{2}-\theta) - i\sin(\frac{\pi}{2}-\theta)}\right]^n$ 6

7 $= \left[\frac{1+\cos A + i\sin A}{1+\cos A - i\sin A}\right]^n$ put $\frac{\pi}{2} - \theta = A$

$= \left[\frac{1 + (\cos A + i\sin A)}{1 + (\cos A + i\sin A)^{-1}}\right]^n$ (using De-moivre's thm)

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MATH-III/21

$$= \left[\frac{1+z}{1+z^{-1}}\right]^n \quad \text{put } z = \cos A + i\sin A$$

$$= \left[\frac{1+z}{1+\frac{1}{z}}\right]^n = \left[\frac{(1+z)}{(1+z)/z}\right]^n = \left[(1+z) \times \frac{z}{(1+z)}\right]^n$$

$$= z^n = [\cos A + i\sin A]^n = \cos nA + i\sin nA$$

$$= \cos n\left(\frac{\pi}{2}-\theta\right) + i\sin n\left(\frac{\pi}{2}-\theta\right)$$

$$= \cos\left(\frac{n\pi}{2}-n\theta\right) + i\sin\left(\frac{n\pi}{2}-n\theta\right) = \text{RHS}$$

$\square$

→END←

MATRICES

MATH-211 / 1

RANK OF MATRIX

→ Rank of a matrix is the order of the highest order non-vanishing minor.

→ If A is a matrix then its Rank is denoted as $\rho(A)$.

NOTE (i) Zero matrix has no Rank

(ii) ~~Unit matrix of order n has rank n~~

(iii) nth order non-singular matrix has rank n

(iv) nth order singular matrix has rank < n

(v) ~~If $O(A) = m\times n$ then~~

(a) If $m>n \Rightarrow \rho(A) \le n$

(b) If $m<n \Rightarrow \rho(A) \le m$

(c) If $m=n \Rightarrow \rho(A) \le m$ or n.

ELEMENTARY TRANSFORMATION OF A MATRIX

~~There are three types of Elementary transformation~~

i) Interchange of any two rows (or columns)

ie $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$

(ii) ~~Multiplication of~~ ~~a~~ ~~non zero~~ any row (or column) by a non zero number.

~~ie $C_i \to KC_i$ or $R_i \to KR_i$, $K\neq 0$~~

(iii) ~~Addition of~~ constant multiple of any row (or column) to the corresponding Elements of any other rows (or columns)

~~ie $R_i \Rightarrow R_i + KR_j$, $C_i \to C_i + KC_j$.~~

January 2017

Wk	M	T	W	T	F	S	S
53	30	31					1
1	2	3	4	5	6	7	8
2	9	10	11	12	13	14	15
3	16	17	18	19	20	21	22
4	23	24	25	26	27	28	29

February 2017

Wk	M	T	W	T	F	S	S
5			1	2	3	4	5
6	6	7	8	9	10	11	12
7	13	14	15	16	17	18	19
8	20	21	22	23	24	25	26
9	27	28					

MATH iii /2.

Note that Elementary transformation doesnot Alter rank of the matrix.

Problems:1 Find rank of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Soln Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$O(A) = 3\times 3 \Rightarrow \rho(A) \leq 3$

Now $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$

$= 1\{(5)(9)-(6)(8)\} - 2\{(4)(9)-(7)(6)\} + 3\{(4)(8)-(7)(5)\}$

$= 1\{45-48\} - 2\{36-42\} + 3\{32-35\}$

$= 1\{-3\} - 2\{-6\} + 3\{-3\}$

$= -3+12-9$

$= -12+12$

$= 0$ i.e 3rd order minors vanishes.

$\therefore \rho(A) \neq 3$

~~Now let's consider a 2nd~~

But there exist ($\exists$) a 2nd order non-vanishing minor which is $|A_1| = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5-8 = -3 \neq 0$

Hence $\rho(A) = 2$ (Ans)

MATH-III/3

HOME TASK

No2 Find Rank of following matrices.

(a) $\begin{bmatrix} 0 & 0 & 2 \\ 1 & -1 & 3 \\ 1 & 2 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 0 & 2 \\ 3 & 1 & 4 \\ 2 & 0 & 2 \end{bmatrix}$

Co-efficient and Augmented matrix for system of Linear Equation:

Let's consider system of linear equation

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= k_1 \\ a_2x + b_2y + c_2z &= k_2 \\ a_3x + b_3y + c_3z &= k_3 \end{aligned}\right\} \longrightarrow (1)$$

To test consistency of system (1) we consider the rank of matrices

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad K = \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & k_1 \\ a_2 & b_2 & c_2 & k_2 \\ a_3 & b_3 & c_3 & k_3 \end{array}\right]$$

Here A is called co-efficient matrix & K is called Augmented matrix.

ROUCHE'S THEOREM It states that the system of Equations is consistent if and only if the co-efficient matrix and the Augmented matrix are of same Rank. otherwise the system is inconsistent.

MATH-III / 4

PROCEDURE TO TEST CONSISTENCY:

Step-i

Write down the system of Equation in General form & find out A, K from it.

Step-ii

Find out ranks of A and K by reducing A to triangular form through Elementary transformation.

Let $\rho(A) = \gamma$, $\rho(K) = \gamma'$, No. of variables $= n$

then Apply any one of formula given below:

Step-iii

(a) If $\gamma \neq \gamma'$ ⇒ The system is inconsistent (i.e has no solution)

(b) If $\gamma = \gamma' = n$ ⇒ The system is consistent and has unique sol^n

(c) If $\gamma = \gamma' < n$ ⇒ The system is consistent and has infinite number of sol^n.

In that case set $(n-\gamma)$ No. of variables to arbitrary constants to find sol^n.

MATH iii/5

NO3. Find rank of matrix $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

Soln Let $A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{bmatrix} \qquad \begin{array}{l} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_3 \to R_3 - \tfrac{1}{2} R_2 \qquad (1)$$

From above transfermed matrix we have All 3rd order minor vanishes. Hence $\rho(A) \neq 3$ i.e $\rho(A) < 3$

Now $\exists$ a 2nd order non vanishing minor which is $|A_1| = \begin{vmatrix} -1 & 2 \\ 0 & 8 \end{vmatrix} = -8 - 0$
$= -8 \neq 0$

Hence $\rho(A) = 2$ (Ans).

MATH-iii / 6

NO4) Find rank of $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$

Soln

Let $A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - R_1$

$= \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

$R_3 \to R_3 - \frac{1}{3}R_2$

$= \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ——— (1)

From above matrix we have All 3rd order minor vanishes. Hence $\rho(A) \neq 3$ i.e. $\rho(A) < 3$.

Now $\exists$ a 2nd order nonvanishing minor which is $|A_1| = \begin{vmatrix} 4 & 3 \\ 0 & -6 \end{vmatrix} = -24 - 0$
$= -24 \neq 0$.

Hence $\rho(A) = 2$ (Ans).

MATH-iii/7

NO5 Test consistency of following system & if possible find its solution.

$$4x-2y+6z=8$$
$$x+y-3z=-1$$
$$15x-3y+9z=21$$

SOL^n Given system of equations is

$$\left.\begin{aligned} 4x-2y+6z&=8\\ x+y-3z&=-1\\ 15x-3y+9z&=21\end{aligned}\right\} \longrightarrow (1)$$

Here co-efficient matrix

$$A=\begin{bmatrix}4 & -2 & 6\\ 1 & 1 & -3\\ 15 & -3 & 9\end{bmatrix}$$

Augmented matrix

$$K=\left[\begin{array}{ccc:c}4 & -2 & 6 & 8\\ 1 & 1 & -3 & -1\\ 15 & -3 & 9 & 21\end{array}\right]$$

$$=\left[\begin{array}{ccc:c}1 & 1 & -3 & -1\\ 4 & -2 & 6 & 8\\ 15 & -3 & 9 & 21\end{array}\right] \quad R_1\leftrightarrow R_2$$

$$=\left[\begin{array}{ccc:c}1 & 1 & -3 & -1\\ 0 & -6 & 18 & 12\\ 0 & -18 & 54 & 36\end{array}\right] \quad \begin{aligned}R_2&\to R_2-4R_1\\ R_3&\to R_3-15R_1\end{aligned}$$

MATH-111/8

$$= \left[\begin{array}{ccc:c} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad R_3 \to R_3 - 3R_2 \qquad \text{(2)}$$

For A,
All 3rd order minor vanishes. But there exist a 2nd order non-vanishing minor which is

$$|A_1| = \begin{vmatrix} 1 & 1 \\ 0 & -6 \end{vmatrix} = (1)(-6) - (0)(1) = -6 \neq 0$$

$\therefore \rho(A) = r = 2$

For K, All 3rd order minor vanishes. But there exist a 2nd order ~~non vanish~~ non vanishing minor which is $|A_1| \neq 0$

Hence $\rho(K) = r' = 2$.

Here $n = 3$

$\therefore r = r' < n$

Hence Given system is consistent and has infinite number of solution.

Now $n - r = 3 - 2 = 1$

Hence let $z = a$, Now from (2)

$$0 \cdot x - 6 \cdot y + 18z = 12 \Rightarrow -6y + 18a = 12$$

$$\Rightarrow -y + 3a = 2 \Rightarrow y = 3a - 2$$

Again $x + y - 3z = -1 \Rightarrow x + 3a - 2 - 3a = -1$

$\Rightarrow x = 1$

$\therefore x = 1,\ y = 3a - 2,\ z = a$ is the solution.

MATH-111/9

6) ~~Solve following sy~~

Test the consistency of following system and if possible solve it.

$$2x-3y+7z=5$$
$$3x+y-3z=13$$
$$2x+19y-47z=32$$

<u>soln</u> Given system is

$$\left.\begin{aligned}2x-3y+7z&=5\\3x+y-3z&=13\\2x+19y-47z&=32\end{aligned}\right\} \longrightarrow (1)$$

Co-efficient matrix

$$A=\begin{bmatrix}2 & -3 & 7\\3 & 1 & -3\\2 & 19 & -47\end{bmatrix}$$

Augmented matrix

$$K=\left[\begin{array}{ccc:c}2 & -3 & 7 & 5\\3 & 1 & -3 & 13\\2 & 19 & -47 & 32\end{array}\right]$$

$$=\left[\begin{array}{ccc:c}2 & -3 & 7 & 5\\1 & 4 & -10 & 8\\2 & 19 & -47 & 32\end{array}\right] \quad R_2 \to R_2-R_1$$

Sunday 12

$$=\left[\begin{array}{ccc:c}1 & 4 & -10 & 8\\2 & -3 & 7 & 5\\2 & 19 & -47 & 32\end{array}\right] \quad R_1 \leftrightarrow R_2$$

MATH-iii/10

$$\left[\begin{array}{ccc:c} 1 & 4 & -10 & 8 \\ 0 & -11 & 27 & -11 \\ 0 & 11 & -27 & 16 \end{array}\right] \qquad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$R_3 \to R_3 + R_2$

$$\left[\begin{array}{ccc:c} 1 & 4 & -10 & 8 \\ 0 & -11 & 27 & -11 \\ 0 & 0 & 0 & 5 \end{array}\right] \text{ --- } (2).$$

For A, All 3rd order minor vanishes.
But $\exists$ a 2nd order non-vanishing minor which is $|A_1| = \begin{vmatrix} 1 & 4 \\ 0 & -11 \end{vmatrix} = -11 \neq 0$

$\therefore \rho(A) = r = 2$

For K, $\exists$ a 3rd order non-vanishing minor which is $|A_2| = \begin{vmatrix} 4 & -10 & 8 \\ -11 & 27 & -11 \\ 0 & 0 & 5 \end{vmatrix}$

$= 5\begin{vmatrix} 4 & -10 \\ -11 & 27 \end{vmatrix}$ Expanding w.r.t R_3

$= 5(108 - 110) = -10 \neq 0$

$\therefore \rho(K) = r' = 3$

Here $r \neq r'$

Hence Given system is inconsistent (Ans)

MATH-iii/11

7)) Test consistency of following system and if possible solve it ?

$$x+2y+z=3$$
$$2x+3y+2z=5$$
$$3x-5y+5z=2$$
$$3x+9y-z=4$$

Soln Given system is

$$\left.\begin{aligned} x+2y+z&=3\\ 2x+3y+2z&=5\\ 3x-5y+5z&=2\\ 3x+9y-z&=4 \end{aligned}\right\} \text{———} (1)$$

co-efficient matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & -5 & 5 \\ 3 & 9 & -1 \end{bmatrix}$

Augmented Matrix $K = \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 2 & 3 & 2 & 5 \\ 3 & -5 & 5 & 2 \\ 3 & 9 & -1 & 4 \end{array}\right]$

$= \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & -11 & 2 & -7 \\ 0 & 3 & -4 & -5 \end{array}\right]$ $\quad R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - 3R_1$

$= \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & -4 & -8 \end{array}\right]$ $\quad R_3 \to R_3 - 11R_2$, $R_4 \to R_4 + 3R_2$

2017

Day 046-319 • Week 07

15 February Wednesday

Wk	M	T	W	T	F	S	S
January 2017							
53	30	31					1
1	2	3	4	5	6	7	8
2	9	10	11	12	13	14	15
3	16	17	18	19	20	21	22
4	23	24	25	26	27	28	29
February 2017							
5			1	2	3	4	5
6	6	7	8	9	10	11	12
7	13	14	15	16	17	18	19
8	20	21	22	23	24	25	26
9	27	28					

MATH-III/12.

$R_4 \to R_4 + 2R_3$

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{———} (2)$$

For A, $\exists$ a 3rd order non-vanishing Minor which is $|A_1| = \begin{vmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 1 \begin{vmatrix} -1 & 0 \\ 0 & 2 \end{vmatrix} = -2 \neq 0$

$\therefore \rho(A) = r = 3$

For K $\exists$ a 3rd order nonvanishing Minor which is also $|A_1| = -2 \neq 0$

$\therefore \rho(K) = r' = 3$

Here $n = 3$

$\therefore r = r' = n \Rightarrow$ The system is consistent and has unique solution

From (2) $2z = 4 \Rightarrow z = 2$.

$-y = -1 \Rightarrow y = 1$

$x + 2y + z = 3$

$\Rightarrow x + 2\times 1 + 2 = 3 \Rightarrow x = 3 - 4 = -1$

$\therefore x = -1, y = 1, z = 2$ is the Required solution.

MATH-111/13

8) Investigate for what values of λ & μ the simultaneous equations
$x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=\mu$
have (i) no solution (ii) a unique solution (iii) Infinite number of solution.

Soln Given system is

$$\left.\begin{aligned} x+y+z&=6\\ x+2y+3z&=10\\ x+2y+\lambda z&=\mu \end{aligned}\right\} \text{—— (1)}$$

Co-efficient matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}$$

Augmented matrix $K = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$

$$= \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda-1 & : & \mu-6 \end{bmatrix} \quad \begin{aligned} R_2 &\to R_2 - R_1 \\ R_3 &\to R_3 - R_1 \end{aligned}$$

$$= \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda-3 & : & \mu-10 \end{bmatrix} \quad R_3 \to R_3 - R_2 \quad \text{—— (2)}$$

(i) for $\lambda-3=0$ & $\mu-10\neq0$ i.e for $\lambda=3, \mu\neq10$ the system has no solution.
as $\rho(A)=r=2$, $\rho(K)=r'=3$
Here $r\neq r'$.

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(ii) For $\lambda - 3 \neq 0$ & $\mu - 10 = 0$ or $\mu - 10 \neq 0$ i.e. for $\lambda \neq 3$ & for arbitrary μ the system has a unique solution.

As $\rho(A) = r = 3$, $\rho(K) = r' = 3$; $n = 3$

Here $r = r' = n$.

(iii) For $\lambda - 3 = 0$ & $\mu - 10 = 0$ i.e. for $\lambda = 3$ & $\mu = 10$ the system has infinite solutions

As $\rho(A) = r = 2$, $\rho(K) = r' = 2$, $n = 3$

Here $r = r' < n$.

* Rough for Q.NO 8 (ପରୀକ୍ଷାରେ ଉତ୍ତର ଖାତାରେ ଲେଖିବ ନାହିଁ, Rough page ରେ କରିବ).

i) If $\lambda - 3 = 0$, $\mu - 10 = 0$.

then $\rho(A) = 2$, $\rho(K) = 2$, $n = 3$

$\boxed{\text{Infinite solution}}$

(ii) If $\lambda - 3 \neq 0$, $\mu - 10 \neq 0$

$\rho(A) = r = 3$, $\rho(K) = r' = 3$, $n = 3$

$\boxed{\text{unique soln}}$

(iii) $\lambda - 3 \neq 0$, $\mu - 10 = 0$

$\rho(A) = r = 3$, $\rho(K) = r' = 3$, $n = 3$

$\boxed{\text{unique soln}}$

(iv) $\lambda - 3 = 0$, $\mu - 10 \neq 0$

$\rho(A) = r = 2$, $\rho(K) = 3$

$\boxed{\text{NO soln}}$

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HOME TASK

9)) Find Rank of following matrices.

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$

10)) Test consistency & solve is possible

(a) $2x+6y+11z=0, 6x+20y-6z+3=0,$
$6y-18y+1=0$

(b) $5x+3y+7z=4, 3x+26y+2z=9,$
$7x+2y+10z=5$

(c) $x+5y+7z=15, 2x+3y+4z=11, x-2y-3z$
$=-4, 3x+11y+13z=25.$

(d) $x-2y-z=5, x+8y-3z=-1, 2x+y-3z=7$

11) Investigate the values of λ and μ so that the equations $2x+3y+5z=9,$
$7x+3y-2z=8, 2x+3y+\lambda z=\mu$
(i) has no solution, (ii) has a unique solution
(iii) has infinite number of solution.

January 2017							
Wk	M	T	W	T	F	S	S
53	30	31					1
1	2	3	4	5	6	7	8
2	9	10	11	12	13	14	15
3	16	17	18	19	20	21	22
4	23	24	25	26	27	28	29

February 2017							
Wk	M	T	W	T	F	S	S
5			1	2	3	4	5
6	6	7	8	9	10	11	12
7	13	14	15	16	17	18	19
8	20	21	22	23	24	25	26
9	27	28					

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12)) Show that the equations $3x+4y+5z=a$, $4x+5y+6z=b$, $5x+6y+7z=c$ do not have a solution unless $a+c=2b$

→ End ←

3: LINEAR DIFFERENTIAL EQUATION

MATH-iii/1

(1) Linear differential equation (LDE)

(i) The differential equation in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together.

General form of nth. order LDE is

$$\boxed{\frac{d^n y}{dx^n} + K_1 \frac{d^{n-1} y}{dx^{n-1}} + \text{------} + K_n y = f(x)} \quad \text{---} (1)$$

where $K_1, K_2 \text{---} K_n$ are constants or functions of x.

(ii) Ex: $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$, $\frac{d^2y}{dx^2} - 4y = x^2$

are LDE.

2) Homogeneous LDE with constant co-efficient

(i) write down point (i) of above

(ii) If $f(x) = 0$ and $K_1, K_2 \text{------} K_n$ are constant then Eq^n (1) is called homogeneous LDE with constant co-efficients.

Ex: $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$

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3) Non-Homogeneous LDE with constant co-efficients

(i) write down (i) of sec(1)

(ii) If $f(x) \neq 0$ and $K_1, K_2 \cdots K_n$ are constants then equation ① is called Non-homogeneous LDE with constant co-efficients.

Ex $\frac{d^2y}{dx^2} - 4y = x^2$.

4) OPERATOR D

Let's denote $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2 \cdots \frac{d^n}{dx^n} = D^n$

then $Dy = \frac{dy}{dx}$, $D^2y = \frac{d^2y}{dx^2} \cdots D^ny = \frac{d^ny}{dx^n}$

D is called Differential operator.

Ex $D^2(x^2) = \frac{d^2}{dx^2}(x^2) = \frac{d}{dx}\left(\frac{d}{dx}x^2\right)$

$= \frac{d}{dx}(2x) = 2\frac{d}{dx}(x) = 2 \times 1 = 2$.

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~~INVERSE OPERATOR~~

5) symbolic form of LDE

Let's consider nth order LDE

$$\frac{d^n y}{dx^n} + K_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + K_n y = f(x) \quad \text{—— (1)}$$

$$\Rightarrow \left[\frac{d^n}{dx^n} + K_1 \frac{d^{n-1}}{dx^{n-1}} + \cdots + K_n\right] y = f(x)$$

$$\Rightarrow \left[D^n + K_1 D^{n-1} + \cdots + K_n\right] y = f(x)$$

$$\Rightarrow \left[F(D)\right] y = f(x) \quad \text{—— (2)}$$

which is the symbolic form of (1)

where $F(D) = D^n + K_1 D^{n-1} + \cdots + K_n$

Ex $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2$

$$\Rightarrow \left[D^2 - 5D + 6\right] y = x^2 \quad \text{—— (1)}$$

Here $F(D) = D^2 - 5D + 6$; $f(x) = x^2$.

6) INVERSE OPERATOR:

i) $\frac{1}{F(D)} f(x)$ is that function of x, not containing arbitrary constants which when operated upon by $F(D)$ gives $f(x)$.

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(ii) $\frac{1}{F(D)}$ is called inverse operator.

NOTE

(i) $\frac{1}{D} f(x) = \int f(x)\,dx$

(ii) $\frac{1}{D-a} f(x) = e^{ax} \int e^{-ax} f(x)\,dx$.

NOTE

(i) $(1-x)^{-1} = 1+x+x^2+x^3+\cdots$

(ii) $(1+x)^{-1} = 1-x+x^2-x^3+\cdots$

(iii) $(1-x)^{-2} = 1+2x+3x^2+4x^3+\cdots$

(iv) $(1+x)^{-2} = 1-2x+3x^2-4x^3+\cdots$

(v) $\sinh x = \frac{e^{x}-e^{-x}}{2}$

$\cosh x = \frac{e^{x}+e^{-x}}{2}$

(vi) $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$

$\cos x = \frac{e^{ix}+e^{-ix}}{2}$

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7) SOLUTION OF LDE

Complete solution of a LDE contains two parts
(i) Complementary function (C.F)
(ii) particular integral (P.I)

The complete solution (C.S) is given by

$$\boxed{C.S = C.F + P.I}$$

$$\Rightarrow \boxed{y = C.F + P.I}$$ when Dependent variable is y.

8) Rules for find C.F

Step-i Write down the given LDE in symbolic form as $[F(D)]y = f(x)$ ——— (1)
& find $F(D)$, $f(x)$ from it.

Step-ii Solve the Auxiliary equation for (1)

$$\boxed{F(m) = 0}$$

Let m_1, m_2, m_3 ——— m_n are roots

Step-iii Find C.F according to nature of roots by following formula:

(a) If $m_1, m_2, \ldots\ldots m_n$ are real and different then

$$\boxed{C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots\ldots + c_n e^{m_n x}}$$

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(b) $m_1 = m_2 = m_3 = \cdots m_r = m$ are real & $m_{r+1} \cdots m_n$ are real and different then

$$\boxed{C.F = (c_1 + c_2 x + c_3 x^2 + \cdots + c_r x^{r-1}) e^{mx} + c_{r+1} e^{m_{r+1} x} + \cdots + c_n e^{m_n x}}$$

(c) If a pair of roots are complex conjugate i.e roots $\boxed{m_1 = \alpha + i\beta,\ m_2 = \alpha - i\beta}$ others are real and different. then

$$\boxed{C.F = (c_1 \cos\beta x + c_2 \sin\beta x) e^{\alpha x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}}$$

(d) If two pairs of roots are complex conjugate and equal i.e if roots are $m_1 = m_2 = \alpha + i\beta$, $m_3 = m_4 = \alpha - i\beta$ and others are real & different. then

$\cancel{C.F = (c_1 + c_2 x) e^{\alpha x} +}$

$$\boxed{C.F = [(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x] e^{\alpha x} + c_5 e^{m_5 x} + \cdots + c_n e^{m_n x}}$$

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9)) FORMULA FOR P I

(a) $P.I = \frac{1}{F(D)} f(x)$.

(b) $\frac{1}{D} f(x) = \int f(x)\,dx$

(c) $\frac{1}{D-a} f(x) = e^{ax} \int e^{-ax} f(x)\,dx$

(d) $\frac{1}{F(D)} e^{ax} = \frac{1}{F(a)} e^{ax}, \quad F(a) \neq 0$

$= x \frac{1}{F'(a)} e^{ax}, \quad \text{If } F(a) = 0, F'(a) \neq 0$

$= x^2 \frac{1}{F''(a)} e^{ax}, \quad \text{If } F'(a) = 0, F''(a) \neq 0$

and so on

(e) $\frac{1}{F(D^2)} \sin(ax+b) = \frac{1}{F(-a^2)} \sin(ax+b), \quad F(-a^2) \neq 0$

$= x \frac{1}{F'(-a^2)} \sin(ax+b), \quad F(-a^2) = 0, F'(-a^2) \neq 0$

$= x^2 \frac{1}{F''(-a^2)} \sin(ax+b), \quad F'(-a^2) = 0, F''(-a^2) \neq 0$

and so on

(f) $\frac{1}{F(D^2)} \cos(ax+b)$ same as (e) & replace $\sin(ax+b)$ by $\cos(ax+b)$

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g) $\boxed{\frac{1}{F(D)}(e^{ax}V) = e^{ax}\frac{1}{F(D+a)}V}$

V is a function of x.

(h) $\boxed{\frac{1}{F(D)}(xV) = x\frac{1}{F(D)}V - \frac{F'(D)}{[F(D)]^2}V}$

V is a function of x.

(i) For $\frac{1}{F(D)}f(x)$ we may use partial fractions for $\frac{1}{F(D)}$ and we operate term by term and then we apply the formula (c)

Problems

1) Find complementary function (C.F) for $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$.

Soln $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0 \Rightarrow [D^2 - 5D + 6]y = 0$ — (1)

Here $F(D) = D^2 - 5D + 6$, $f(x) = 0$.

Auxiliary equation for (1) is $F(m) = 0$

$\Rightarrow m^2 - 5m + 6 = 0 \Rightarrow m = \frac{-(-5) \pm \sqrt{(-5)^2 - 4\cdot 1\cdot 6}}{2\times 1} = \frac{5 \pm 1}{2} =$

$\frac{5+1}{2}$ or $\frac{5-1}{2} = 3, 2$.

$\therefore$ Roots are 2, 3

Hence $C.F = c_1e^{2x} + c_2e^{3x}$ (Ans)

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(2) Solve $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$

Soln $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$

$\Rightarrow [D^3 - 3D^2 + 3D - 1]y = 0$ ——— (1) $\frac{d^n}{dx^n} \equiv D^n$

Here $F(D) = D^3 - 3D^2 + 3D - 1$, $f(x) = 0$

Auxilliary Equation for (1) is

$F(m) = 0 \Rightarrow m^3 - 3m^2 + 3m - 1 = 0$

$\Rightarrow (m)^3 - 3(m)^2(1) + 3(m)(1)^2 - (1)^3 = 0$

$\Rightarrow (m-1)^3 = 0 \Rightarrow m = 1, 1, 1$.

$\therefore$ Roots are $1, 1, 1$

Hence $C.F = (c_1 + c_2x + c_3x^2)e^x$ ——— (2)

Now $P.I = \frac{1}{F(D)} f(x) = \frac{1}{F(D)}(0) = 0$ ——— (3)

$C.S = C.F + P.I$

$\Rightarrow y = (c_1 + c_2x + c_3x^2)e^x + 0$

$\Rightarrow y = (c_1 + c_2x + c_3x^2)e^x$ (Ans)

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(3) Solve $\frac{d^4x}{dt^4}+4x=0$

Sol^n $\frac{d^4x}{dt^4}+4x=0$

$\Rightarrow (D^4+4)x=0$ —— (1) $\quad D^n=\frac{d^n}{dt^n}$

Here $F(D)=D^4+4$, $f(t)=0$

Auxiliary Equation for (1) is

$F(m)=0$

$\Rightarrow$ [illegible] $m^4+4=0$

$\Rightarrow (m^2)^2+2.(m^2).2+(2)^2$ [illegible] $-4m^2=0$

$\Rightarrow (m^2+2)^2-4m^2=0$

$\Rightarrow (m^2+2)^2-(2m)^2=0$

$\Rightarrow (m^2+2+2m)(m^2+2-2m)=0$

$\Rightarrow m^2+2m+2=0$ or $m^2-2m+2=0$

$\Rightarrow m=\frac{-2\pm\sqrt{(2)^2-4.1.2}}{2\times1}$ or $m=\frac{-(-2)\pm\sqrt{(-2)^2-4.1.2}}{2\times1}$

$\Rightarrow m=\frac{-2\pm\sqrt{4-8}}{2}$ or $m=\frac{2\pm\sqrt{4-8}}{2}$

$\Rightarrow m=\frac{-2\pm\sqrt{-4}}{2}$ or $m=\frac{2\pm\sqrt{-4}}{2}$

Scanned with CamScanner

Wk	M	T	W	T	F	S	S
13						1	2
14	3	4	5	6	7	8	9
15	10	11	12	13	14	15	16
16	17	18	19	20	21	22	23
17	24	25	26	27	28	29	30

Wk	M	T	W	T	F	S	S
18	1	2	3	4	5	6	7
19	8	9	10	11	12	13	14
20	15	16	17	18	19	20	21
21	22	23	24	25	26	27	28
22	29	30	31				

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$m = \frac{-2 \pm 2i}{2}$ or $\frac{2 \pm 2i}{2}$

$\Rightarrow m = \frac{2(-1 \pm i)}{2}$ or $\frac{2(1 \pm i)}{2}$

$\Rightarrow m = -1 \pm i$ or $m = 1 \pm i$

$\therefore$ Roots are $-1+i, -1-i, 1+i, 1-i$

Hence $C.F = (c_1 \cos t + c_2 \sin t)e^{-t} + (c_3 \cos t + c_4 \sin t)e^{t}$

Now $P.I = \frac{1}{F(D)} f(t) = \frac{1}{F(D)}(0) = 0$

Now complete solution

$C.S = C.F + P.I$

$\Rightarrow x = (c_1 \cos t + c_2 \sin t)e^{-t} + (c_3 \cos t + c_4 \sin t)e^{t} + 0$

$\Rightarrow x = (c_1 \cos t + c_2 \sin t)e^{-t} + (c_3 \cos t + c_4 \sin t)e^{t}$ (Ans)

(4) Solve $[D^3 - 6D^2 + 11D - 6]y = e^{-2x} + e^{-3x}$

Soln: $[D^3 - 6D^2 + 11D - 6]y = e^{-2x} + e^{-3x}$ —(1)

Here $F(D) = D^3 - 6D^2 + 11D - 6$, $f(x) = e^{-2x} + e^{-3x}$

A.E for (1) is $f(m) = 0$

$\Rightarrow m^3 - 6m^2 + 11m - 6 = 0$

$\Rightarrow (m-1)(m^2 - 5m + 6) = 0$

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$\Rightarrow m-1=0$ or $m^2-5m+6=0$

$\Rightarrow m=1$ or $m=\dfrac{-(-5)\pm\sqrt{(-5)^2-4\cdot1\cdot6}}{2\times1}$

$\Rightarrow m=1$ or $m=\dfrac{5\pm\sqrt{25-24}}{2}$

$\Rightarrow m=1$ or $m=\dfrac{5\pm\sqrt{1}}{2}$

$\Rightarrow m=1$ or $m=\dfrac{5\pm1}{2}$

$\Rightarrow m=1$ or $m=\dfrac{5+1}{2}$ or $m=\dfrac{5-1}{2}$

$\Rightarrow m=1$ or $m=3$ or $m=2$.

$\therefore$ Roots are 1, 2, 3

$\therefore CF = c_1e^{x}+c_2e^{2x}+c_3e^{3x}$

Now $P.I = \dfrac{1}{F(D)}f(x)$

$$=\frac{1}{D^3-6D^2+11D-6}\left(e^{-2x}+e^{-3x}\right)$$

$$=\frac{1}{D^3-6D^2+11D-6}e^{-2x}+\frac{1}{D^3-6D^2+11D-6}e^{-3x}$$

$$=\frac{1}{(-2)^3-6(-2)^2+11(-2)-6}e^{-2x}+\frac{1}{(-3)^3-6(-3)^2+11(-3)-6}e^{-3x}$$

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$$= \frac{1}{-60}e^{-2x} + \frac{1}{-120}e^{-3x} = -\frac{1}{60}e^{-2x} - \frac{1}{120}e^{-3x}$$

$\therefore$ C.S = C.F + P.I

$$\Rightarrow y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x} - \frac{1}{60}e^{-2x} - \frac{1}{120}e^{-3x}$$ (Ans)

⑤ Find P.I of $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4\cos^2 x$

Soln $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4\cos^2 x$

$\Rightarrow [D^2 + 3D + 2]y = 4\cos^2 x$ —— ① $D^n = \frac{d^n}{dx^n}$

Here $F(D) = D^2 + 3D + 2$, $f(x) = 4\cos^2 x$

$$P.I = \frac{1}{F(D)} f(x) = \frac{1}{D^2+3D+2}(4\cos^2 x)$$

$$= \frac{1}{D^2+3D+2}[2\cdot(2\cos^2 x)]$$

$$= \frac{1}{D^2+3D+2}[2\cdot(1+\cos 2x)]$$

$$= \frac{1}{D^2+3D+2}[2 + 2\cos 2x]$$

$$= \frac{1}{D^2+3D+2}(2) + \frac{1}{D^2+3D+2} 2\cos 2x$$

$$= 2\frac{1}{D^2+3D+2}(1) + 2\frac{1}{D^2+3D+2}(\cos 2x)$$

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$$= 2\frac{1}{D^2+3D+2}(e^{0\times x}) + 2\cdot\frac{1}{-2^2+3D+2}\cos 2x$$

$$= 2\times\frac{1}{0^2+3\times 0+2}(1) + \frac{2}{3D-2}\cos 2x$$

$$= 1 + \frac{2(3D+2)}{(3D-2)(3D+2)}\cos 2x$$

$$= 1 + \frac{2(3D+2)}{9D^2-4}\cos 2x$$

$$= 1 + \frac{2(3D+2)}{9(-2^2)-4}\cos 2x$$

$$= 1 + \frac{2(3D+2)}{-36-4}\cos 2x$$

$$= 1 - \frac{1}{20}(3D+2)\cos 2x$$

$$= 1 - \frac{1}{20}\left[-6\sin 2x + 2\cos 2x\right]$$

$$= 1 + \frac{1}{10}\left[3\sin 2x - \cos 2x\right] \text{ (Ans)}$$

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6) Solve $(D^2-4D+3)y = \sin 3x \cdot \cos 2x$

Soln $(D^2-4D+3)y = \sin 3x \cdot \cos 2x$ ——— (1)

Here $F(D) = D^2-4D+3$, $f(x) = \sin 3x \cdot \cos 2x$

A.E for (1) is $F(m) = 0$

$\Rightarrow m^2-4m+3=0 \Rightarrow m = \dfrac{-(-4) \pm \sqrt{(-4)^2-4\cdot 1\cdot 3}}{2\times 1}$

$\Rightarrow m = \dfrac{4 \pm \sqrt{16-12}}{2} \Rightarrow m = \dfrac{4 \pm \sqrt{4}}{2} \Rightarrow m = \dfrac{4\pm 2}{2}$

$\Rightarrow m = \dfrac{4+2}{2}$ or $m = \dfrac{4-2}{2} \Rightarrow m = 3$ or 1

$\therefore$ Roots are 1, 3.

Hence $C.F = c_1 e^{x} + c_2 e^{3x}$ ——— (2)

Now $P.I = \dfrac{1}{F(D)} f(x) = \dfrac{1}{D^2-4D+3} (\sin 3x \cdot \cos 2x)$

$= \dfrac{1}{2} \times \dfrac{1}{D^2-4D+3} (2 \sin 3x \cos 2x)$

$= \dfrac{1}{2} \times \dfrac{1}{D^2-4D+3} \{\sin(3x+2x) + \sin(3x-2x)\}$

$= \dfrac{1}{2} \times \dfrac{1}{D^2-4D+3} (\sin 5x + \sin x)$

$= \dfrac{1}{2} \times \dfrac{1}{D^2-4D+3} \sin 5x + \dfrac{1}{2} \times \dfrac{1}{D^2-4D+3} \sin x$

$= P.I_1 + P.I_2$ ——— (3) (say)

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$$P.I_1 = \frac{1}{2} \times \frac{1}{D^2-4D+3}(\sin 5x)$$

$$= \frac{1}{2} \times \frac{1}{-5^2-4D+3}\sin 5x$$

$$= \frac{1}{2} \times \frac{1}{-25-4D+3}\sin 5x$$

$$= \frac{1}{2} \times \frac{1}{-4D-22}(\sin 5x)$$

$$= -\frac{1}{4} \times \frac{1}{2D+11}(\sin 5x) \checkmark$$

$$= -\frac{1}{4} \times \frac{2D-11}{(2D+11)(2D-11)}(\sin 5x)$$

$$= -\frac{1}{4} \times \frac{(2D-11)}{4D^2-121}(\sin 5x)$$

$$= -\frac{1}{4} \times \frac{(2D-11)}{4\cdot(-5^2)-121}(\sin 5x)$$

$$= -\frac{1}{4} \frac{2D-11}{-100-121}(\sin 5x)$$

$$= \frac{1}{884}(2D-11)(\sin 5x)$$

$$= \frac{1}{884}(10\cos 5x - 11\sin 5x)$$

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$$P.I_2 = \frac{1}{2} \times \frac{1}{D^2-4D+3} \sin x$$

$$= \frac{1}{2} \times \frac{1}{-1^2-4D+3} \sin x = \frac{1}{2} \times \frac{1}{-4D+2} \sin x$$

$$= -\frac{1}{4} \times \frac{1}{(2D-1)} \sin x = -\frac{1}{4} \times \frac{2D+1}{(2D-1)(2D+1)} \sin x$$

$$= -\frac{1}{4} \times \frac{(2D+1)}{4D^2-1} (\sin x) = -\frac{1}{4} \times \frac{(2D+1)}{4(-1^2)-1} \sin x$$

$$= -\frac{1}{4} \times \frac{2D+1}{(-5)} (\sin x) = \frac{1}{20} (2D+1)(\sin x)$$

$$= \frac{1}{20} (2\cos x + \sin x)$$

$$\therefore P.I = P.I_1 + P.I_2$$

$$= \frac{1}{884} (10\cos 5x - 11\sin 5x) + \frac{1}{20} (2\cos x + \sin x)$$

Now $C.S = C.F + P.I$

$$\Rightarrow y = c_1 e^{x} + c_2 e^{3x} + \frac{1}{884} (10\cos 5x - 11 \sin 5x) + \frac{1}{20} (2\cos x + \sin x) \quad \text{—(4)}$$

(Ans)

No7) Solve $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x} \sin 2x$

Soln $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x} \sin 2x$

$$\Rightarrow [D^2 + 5D + 6]y = e^{-2x} \sin 2x \quad \text{—(1)}$$

Here $F(D) = D^2 + 5D + 6$, $f(x) = e^{-2x} \sin 2x$.

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February 2017

Wk	M	T	W	T	F	S	S
5			1	2	3	4	5
6	6	7	8	9	10	11	12
7	13	14	15	16	17	18	19
8	20	21	22	23	24	25	26
9	27	28					

March 2017

Wk	M	T	W	T	F	S	S
9			1	2	3	4	5
10	6	7	8	9	10	11	12
11	13	14	15	16	17	18	19
12	20	21	22	23	24	25	26
13	27	28	29	30	31		

MATH-iii-18

A:E for eqn(1) is $F(m)=0 \Rightarrow m^2+5m+6=0$

9 $\Rightarrow m^2+3m+2m+6=0 \Rightarrow m(m+3)+2(m+3)=0$

$\Rightarrow (m+3)(m+2)=0 \Rightarrow m+3=0$ or $m+2=0 \Rightarrow m=-3$ or -2

10 $\therefore$ Roots are $-2, -3$.

Hence $C.F = c_1e^{-2x}+c_2e^{-3x}$ —— (2)

11

$$P.I = \frac{1}{F(D)} f(x) = \frac{1}{D^2+5D+6}(e^{-2x}\sin 2x)$$

12

$$= e^{-2x}\frac{1}{(D-2)^2+5(D-2)+6}\sin 2x$$

1

$$= e^{-2x}\frac{1}{D^2-4D+4+5D-10+6}\sin 2x$$

2

$$= e^{-2x}\frac{1}{D^2+D}(\sin 2x)$$

3

$$= e^{-2x}\frac{1}{-2^2+D}\sin 2x = e^{-2x}\frac{1}{D-4}\sin 2x$$

4

$$= e^{-2x}\frac{(D+4)}{(D-4)(D+4)}\sin 2x = e^{-2x}\frac{(D+4)}{D^2-16}\sin 2x$$

5

$$= e^{-2x}\frac{D+4}{-2^2-16}(\sin 2x) = -\frac{1}{20}(D+4)\sin 2x$$

6

$$= -\frac{1}{20}(2\cos 2x+4\sin 2x)$$

7

$$= -\frac{1}{20}\times 2(\cos 2x+2\sin 2x)$$

$$= -\frac{1}{10}(\cos 2x+2\sin 2x) \quad \text{—— (3)}$$

Complete solution $C.S = C.F + P.I$

$$\Rightarrow y = c_1e^{-2x}+c_2e^{-3x}-\frac{1}{10}(\cos 2x+2\sin 2x)$$

MATH-iii/19

NO 8)) Solve $\frac{d^2y}{dx^2} + 4y = x\sin x$

Soln $\frac{d^2y}{dx^2} + 4y = x\sin x$

$\Rightarrow [D^2+4]y = x\sin x$ ——— (1)

Here $F(D) = D^2+4$, $f(x) = x\sin x$

Now A.E for eqn (1) is $F(m)=0 \Rightarrow m^2+4=0$

$\Rightarrow m^2 = -4 \Rightarrow m = \sqrt{-4} = \pm 2i$

$\therefore$ Roots are $2i, -2i$

Hence $C.F = (C_1\cos 2x + C_2\sin 2x)e^{0\times x} = (C_1\cos 2x + C_2\sin 2x)$

$P.I = \frac{1}{F(D)} f(x) = \frac{1}{D^2+4}(x\sin x)$ ——— (2)

$= x\frac{1}{D^2+4}\sin x - \frac{2D}{(D^2+4)^2}(\sin x)$

$= x\frac{1}{-1^2+4}\sin x - \frac{2D}{(-1^2+4)^2}\sin x$

$= x\frac{1}{3}\sin x - \frac{2D}{9}\sin x$

$= \frac{1}{3}x\sin x - \frac{2D}{9}\sin x$

$= \frac{1}{3}x\sin x - \frac{2}{9}\cos x$ ——— (3)

$\therefore C.S = C.F + P.I$

$\Rightarrow y = (C_1\cos 2x + C_2\sin 2x) + \frac{1}{3}x\sin x - \frac{2}{9}\cos x$

(Ans)

MATH-III/20

Q) Solve $\frac{d^2y}{dx^2} - 4y = x\sinh x$

Soln $\frac{d^2y}{dx^2} - 4y = x\sinh x$

$\Rightarrow (D^2-4)y = x\sinh x$ —— (1)

Here $F(D) = D^2-4$, $f(x) = x\sinh x$

$= x\left(\frac{e^x - e^{-x}}{2}\right)$

A.E for (1) is $F(m)=0 \Rightarrow m^2-4=0$

$\Rightarrow m^2 = 4 \Rightarrow m = \pm 2$

$\therefore$ Roots are $2, -2$

Hence $C.F = c_1e^{2x} + c_2e^{-2x}$ —— (2)

$P.I = \frac{1}{F(D)} f(x) = \frac{1}{D^2-4} \frac{x(e^x - e^{-x})}{2}$

$= \frac{1}{2} \frac{1}{D^2-4} x(e^x - e^{-x})$

$= \frac{1}{2}\left\{x \frac{1}{D^2-4}(e^x - e^{-x}) - \frac{2D}{(D^2-4)^2}(e^x - e^{-x})\right\}$

$= \frac{1}{2}x \frac{1}{D^2-4} e^x - \frac{1}{2}x \frac{1}{D^2-4} e^{-x} - \frac{D}{(D^2-4)^2} e^x$
$+ \frac{D}{(D^2-4)^2} e^{-x}$

$= \frac{1}{2}x \frac{1}{1^2-4} e^x - \frac{1}{2}x \frac{1}{(-1)^2-4} e^{-x} - \frac{D}{\{1^2-4\}^2} e^x$
$+ \frac{D}{\{(-1)^2-4\}^2} e^{-x}$

MATH-III /21

$$= -\frac{1}{6}xe^{x} + \frac{1}{6}xe^{-x} - \frac{D}{9}e^{x} + \frac{D}{9}e^{-x}$$

$$= -\frac{1}{6}xe^{x} + \frac{1}{6}xe^{-x} - \frac{1}{9}e^{x} - \frac{1}{9}e^{-x}$$

$$= -\frac{1}{6}x(e^{x} - e^{-x}) - \frac{1}{9}(e^{x} + e^{-x})$$

$$= -\frac{1}{3}x\left(\frac{e^{x} - e^{-x}}{2}\right) - \frac{2}{9}\left(\frac{e^{x} + e^{-x}}{2}\right)$$

$$= -\frac{1}{3}x\sinh x - \frac{2}{9}\cosh x \qquad (3)$$

C.S = C.F + P.I

$$\Rightarrow y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{3}x\sinh x - \frac{2}{9}\cosh x$$ (Ans)

HOME TASK

(1) solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$

(2) solve $4y''' + 4y'' + y' = 0$

(3) solve $[D^2 + 6D + 9]y = 5e^{3x}$

(4) solve $\frac{d^2y}{dx^2} + 4y = e^{2x}$

(5) solve $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{2x} + \sin 2x$

MATH-III/22.

6) Solve $(D^3-D)y = 2x+1+4\cos x+2e^x$

(7) Solve $(D^4-1)y = e^x\cos x$

(8) Solve $(D^2-4D+4)y = 8x^2e^{2x}\sin 2x$

(9) Simplify following:

(a) $\frac{1}{D^2-4}(\sin 3x)$

(b) $\frac{1}{D}(e^{2x})$

(c) $\frac{1}{D^2+4}(\sin 2x)$

PARTIAL DIFFERENTIAL EQUATION (PDE)

i) The differential equation which involves two or more than two independent variables and partial differential co-efficients w.r.t any of them is called PDE

(ii) Ex $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$

$$\frac{\partial^2 z}{\partial x^2} + z = 0$$

MATH-III / 23

Notations

If $z = f(x,y)$ then

$$p = \frac{\partial z}{\partial x}$$

$$q = \frac{\partial z}{\partial y}$$

$$r = \frac{\partial^2 z}{\partial x^2}$$

$$s = \frac{\partial^2 z}{\partial x \partial y}$$

$$t = \frac{\partial^2 z}{\partial y^2}$$

Note
(i) For $\frac{\partial z}{\partial x}$ (take y as constant)

(ii) For $\frac{\partial z}{\partial y}$ (take x as constant)

(iii) $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$, $\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right).$$

Formation of P.D.E.

A partial differential equation can be formed either by eliminating arbitrary constants or by eliminating arbitrary functions involved in the equation.

MATH-III/24

19)) Form a partial differential equation corresponding to $z = ax + by + a^2 + b^2$

Soln $z = ax + by + a^2 + b^2$ ——— (1)

$p = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(ax + by + a^2 + b^2)$

$= \frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial x}(by) + \frac{\partial}{\partial x}a^2 + \frac{\partial}{\partial x}b^2$

$= a.\frac{\partial}{\partial x}(x) + b.\frac{\partial}{\partial x}(y) + 0 + 0$

$= a \times 1 + b \times 0 + 0 + 0$

$= a$

$\therefore p = a$ ——— (2)

$q = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(ax + by + a^2 + b^2)$

$= a \times 0 + b \times 1 + 0 + 0$

$= b$

$\therefore q = b$ ——— (3)

using (2) & (3) in (1) we have

$z = ax + by + a^2 + b^2$

$\Rightarrow z = px + qy + p^2 + q^2$

which is the required PDE corresponding to given equation (Ans)

MATH-III/25

20) Form a partial differential equation corresponding to $z = f(x^2-y^2)$.

Soln $z = f(x^2-y^2)$ ——— (1)

$p = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x^2-y^2) = f'(x^2-y^2) \cdot \frac{\partial}{\partial x}(x^2-y^2)$

$= f'(x^2-y^2) \cdot 2x$

$\therefore p = 2x \cdot f'(x^2-y^2)$ ——— (2)

Again $q = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y} f(x^2-y^2) = f'(x^2-y^2) \frac{\partial}{\partial y}(x^2-y^2)$

$= f'(x^2-y^2) \cdot (-2y)$

$\Rightarrow q = -2y f'(x^2-y^2)$ ——— (3)

Now $\frac{p}{q} = \frac{2x f'(x^2-y^2)}{-2y f'(x^2-y^2)} = -\frac{x}{y}$

$\Rightarrow py = -qx \Rightarrow py + qx = 0$ which is the required partial differential equation corresponding to given equation. (Ans)

(21) Form a PDE corresponding to $(x-a)^2+(y-b)^2+z^2=c^2$.

Soln $(x-a)^2+(y-b)^2+z^2 = c^2$ ⟶ (1)

• Differentiating (1) w.r.t. x

$\frac{\partial}{\partial x}\{(x-a)^2+(y-b)^2+z^2\} = \frac{\partial}{\partial x}(c^2)$

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$\Rightarrow 2(x-a) + 0 + 2z.\frac{\partial z}{\partial x} = 0$

$\Rightarrow 2(x-a) + 2z.p = 0$

$\Rightarrow (x-a) + z.p = 0$

$\Rightarrow (x-a) = -zp$ —— (2)

Again differentiating (1) w.r.t y we have

$0 + 2.(y-b) + 2z.\frac{\partial z}{\partial y} = 0$

$\Rightarrow (y-b) + z.q = 0$

$\Rightarrow y-b = -zq$ —— (3)

using (2) & (3) in (1).

$(x-a)^2 + (y-b)^2 + z^2 = c^2$

$\Rightarrow (-zp)^2 + (-zq)^2 + z^2 = c^2$

$\Rightarrow z^2p^2 + z^2q^2 + z^2 = c^2$

$\Rightarrow (p^2+q^2+1)z^2 = c^2$ which is the required PDE corresponding to given equation.

22)) Find Differential equation of all spheres whose centre lies on z-axis.

Sol^n We know that Equation of sphere with centre at (a,b,c) and radius r is $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$.

MATH-III/27

Since centre lies on z-axis $a=0, b=0$.
Hence Equation of sphere is

$$x^2+y^2+(z-c)^2 = r^2 \quad \text{——— (1)}$$

Differentiating (1) w.r.t x we have

$$2x+0+2(z-c).\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow x+(z-c)p = 0$$

$$\Rightarrow (z-c)p = -x \quad \text{——— (2)}$$

Again Differentiating (1) w.r.t y we have

$$0+2y+2(z-c)\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow 2y + 2(z-c)q = 0$$

$$\Rightarrow y+(z-c)q = 0$$

$$\Rightarrow (z-c)q = -y \quad \text{——— (3)}$$

Dividing (2) by (3)

$$\frac{(z-c)p}{(z-c)q} = \frac{-x}{-y}$$

$$\Rightarrow \frac{p}{q} = \frac{x}{y}$$

$\Rightarrow py = qx \Rightarrow py-qx=0$ which is the required partial differential equation corresponding to given equation.

02			February 2017				
Wk	M	T	W	T	F	S	S
5			1	2	3	4	5
6	6	7	8	9	10	11	12
7	13	14	15	16	17	18	19
8	20	21	22	23	24	25	26
9	27	28					

03			March 2017				
Wk	M	T	W	T	F	S	S
9			1	2	3	4	5
10	6	7	8	9	10	11	12
11	13	14	15	16	17	18	19
12	20	21	22	23	24	25	26
13	27	28	29	30	31		

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Home Task

23) Form partial differential equation corresponding to

10 (a) $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

11 (b) $z = f(x^2+y^2)$

12 (c) $z = yf(x) + xg(y)$

1 (d) $z = a(x+y) + b(x-y) + abt + c$

2 (e) Find differential equation of all spheres of fixed radius showing their centres

3 in the xy plane.

4 (f) $z = f(x+at) + g(x-at)$

5 (g) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

6

7

LAGRANGE'S LINEAR PARTIAL DIFFERENTIAL EQUATION.

(i) Equation of the form $\boxed{Pp + Qq = R}$ is called Lagrange's linear partial differential equation. Where P, Q, R are functions of x, y, z; $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

(ii) $xp + yq = 3z$ is a Lagrange Linear PDE.
Here $P = x$, $Q = y$, $R = 3z$.

NOTE (i) If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} \Rightarrow \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{la + mc + ne}{lb + md + nf}$

l, m, n are called multipliers.

(ii) If $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a+b}{a-b} = \frac{c+d}{c-d}$ (Componendo & dividendo)

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MATH 111/30

Procedure to solve:

9 **Step-i** Write down the given PDE in the form of

$$Pp + Qq = R \quad \text{——} ①$$

and find P, Q, R from

10 it.

Step-ii: Find subsidiary equation for ① by

11

$$\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}} \quad \text{——} ②$$

12

Step-iii: Considering any two ratios or all at a

1 time from eqn(2), find two integral or solution of Eqn (2). Let $u(x,y,z) = c_1$ & $v(x,y,z) = c_2$ are

2 two integral of Eqn(2).

Step-iv Find solution of ① by $\boxed{\phi(u,v) = 0 \text{ or } u = \phi(v)}$

(24) Solve $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$.

Soln Given equation is $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$ ——①

5 Here $P = \sqrt{x}$, $Q = \sqrt{y}$, $R = \sqrt{z}$.

Now subsidiary equation for ① is

6

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}} \quad \text{——} ②$$

7 From (2)

$$\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} \Rightarrow \int \frac{dx}{\sqrt{x}} = \int \frac{dy}{\sqrt{y}} + 2c_1$$

$$\Rightarrow \int x^{-\frac{1}{2}}dx = \int y^{-\frac{1}{2}}dy + 2c_1$$

$$\Rightarrow \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + 2c_1 \Rightarrow 2\sqrt{x} = 2\sqrt{y} + 2c_1$$

$$\Rightarrow \sqrt{x} - \sqrt{y} = c_1 = u \text{ (say)} \quad \text{——} ③$$

MATH-III/31

From (2) $\frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}} \Rightarrow \int \frac{dy}{\sqrt{y}} = \int \frac{dz}{\sqrt{z}} + 2c_2$

$\Rightarrow 2\sqrt{y} = 2\sqrt{z} + 2c_2 \Rightarrow \sqrt{y} - \sqrt{z} = c_2 = v$ (say) —— (4)

Now soln of (1) is $\phi(u, v) = 0 \Rightarrow \phi(\sqrt{x} - \sqrt{y}, \sqrt{y} - \sqrt{z}) = 0$ (Ans)

(25) Solve $x(y^2 - z^2)p + y(z^2 - x^2)q - z(x^2 - y^2) = 0$

Soln Given Equation is $x(y^2 - z^2)p + y(z^2 - x^2)q - z(x^2 - y^2) = 0$

$\Rightarrow x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$ —— (1)

Here $P = x(y^2 - z^2)$, $Q = y(z^2 - x^2)$, $R = z(x^2 - y^2)$.

Subsidiary equation for (1) is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$\Rightarrow \frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)}$ —— (2)

Now using Multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ in (2) $\Rightarrow$

Each ratio of (2) $= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$

$\Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

$\Rightarrow \int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = \int 0 \cdot dx + \ln c_1$

$\Rightarrow \ln x + \ln y + \ln z = \ln c_1 \Rightarrow \ln(xyz) = \ln c_1$

$\Rightarrow xyz = c_1 = u$ —— (3)

Again using Multipliers x, y, z in (2)

Each ratio of (2) $= \frac{x\,dx + y\,dy + z\,dz}{0}$

$\Rightarrow x\,dx + y\,dy + z\,dz = 0 \Rightarrow \cancel{\int x\,dx + \int y\,dy + \int z\,d}$

$\Rightarrow \int x\,dx + \int y\,dy + \int z\,dz = \int 0 \cdot dx + \frac{c_2}{2}$

$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_2}{2} \Rightarrow x^2 + y^2 + z^2 = c_2 = v$ —— (4)

Soln of (1) is $u = \phi(v)$

$\Rightarrow xyz = \phi(x^2 + y^2 + z^2)$ (Ans)

May 2017 Wk	M	T	W	T	F	S	S
18	1	2	3	4	5	6	7
19	8	9	10	11	12	13	14
20	15	16	17	18	19	20	21
21	22	23	24	25	26	27	28
22	29	30	31				

June 2017 Wk	M	T	W	T	F	S	S
22				1	2	3	4
23	5	6	7	8	9	10	11
24	12	13	14	15	16	17	18
25	19	20	21	22	23	24	25
26	26	27	28	29	30		

WA

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(26) Solve $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

Hints: use Multipliers $\langle \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \rangle$ & $\langle \frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2} \rangle$

(27) Solve $\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$

Hints: use 2nd & 3rd ratio, Multipliers $\langle x, y, z \rangle$

(28) Solve $x(y-z)p + y(z-x)q = z(x-y)$

Hints $\langle \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \rangle$, $\langle 1, 1, 1 \rangle$

(29) Solve $p \tan x + q \tan y = \tan z$.

(30) Solve $p - q = \log(x+y)$

(31) Solve $p\cos(x+y) + q\sin(x+y) = z$

(32) Solve $y^2 p - xyq = x(z-2y)$

(33) Solve $px(z-2y^2) = (z-qy)(z-y^2-2x^3)$.

→END←

4. LAPLACE TRANSFORM

MATH-iii/1

Gamma function

(i) Gamma function is denoted by $\Gamma(n)$ and defined as $\boxed{\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx}$

where n is a positive real number (not necessarily an integer)

(ii) This integral is also known as Eulers integral of second kind.

some deductions

(i) $\Gamma(n+1) = n\Gamma(n)$

(ii) $\Gamma(n+1) = \underline{|n}$

(iii) $\Gamma(1) = 1$

(iv) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Proof (i) We know that $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$ —— (1)

using (1)

$$\Gamma(n+1) = \int_0^\infty x^{(n+1)-1} e^{-x} dx$$

$$= \int_0^\infty x^n . e^{-x} dx$$

$$= \left[x^n \int e^{-x} dx - \int \left(\int e^{-x} dx\right) \times \left(\frac{d}{dx} x^n\right) dx\right]_0^\infty$$

$$= \left[-x^n e^{-x} - \int (-e^{-x}) \times n . x^{n-1} dx\right]_0^\infty$$

MATH-III/2

$$= \left[-x^n e^{-x} + n\int x^{n-1} e^{-x}\, dx\right]_0^\infty$$

$$= \left[-x^n e^{-x}\right]_0^\infty + n\int_0^\infty x^{n-1} e^{-x}\, dx$$

$$= 0 + n\Gamma(n)$$

$$= n\Gamma(n) \quad \square$$

(ii) We know that

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx \quad \text{--- (1)}$$

using (1)

$$\Gamma(n+1) = \int_0^\infty x^{(n+1)-1} e^{-x}\, dx$$

$$= \int_0^\infty x^n \cdot e^{-x}\, dx$$

$$= \left[x^n \int e^{-x} dx - \int \left(\int e^{-x} dx\right) \times \left(\frac{d}{dx} x^n\right) dx\right]_0^\infty$$

$$= \left[-x^n e^{-x} - \int -e^{-x} \times n \cdot x^{n-1}\, dx\right]_0^\infty$$

$$= \left[-x^n e^{-x} + n\int x^{n-1} e^{-x}\, dx\right]_0^\infty$$

$$= \left[-x^n e^{-x}\right]_0^\infty + n\int_0^\infty x^{n-1}\cdot e^{-x}\,dx$$

$$= 0 + n\Gamma(n)$$

$$\therefore \Gamma(n+1) = n\Gamma(n) \quad \text{——— (2)}$$

$$= n\,\Gamma\{(n-1)+1\}$$

$$= n(n-1)\Gamma(n-1) \quad \text{using (2) ——— (3)}$$

$$= n(n-1)\cdot\Gamma\{(n-2)+1\}$$

$$= n(n-1)(n-2)\,\Gamma(n-2) \quad \text{——— (4)}$$

proceeding in this way n times we have

$$= n(n-1)(n-2)\cdots\cdots 1.\,\Gamma(1)$$

$$= \underline{|n}\cdot 1 = \underline{|n} \quad \square$$

(III) We know that $\Gamma(n) = \int_0^\infty e^{-x}\cdot x^{n-1}\,dx$ ——— (1)

using (1) $\Gamma(1) = \int_0^\infty e^{-x}\, x^{1-1}\,dx$

$$= \int_0^\infty e^{-x}\cdot 1\cdot dx$$

$$= \left[-e^{-x}\right]_0^\infty = 0-(-1)$$

$$= 1 \quad \square$$

(iv) We know that $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx$ —— (1)

using Eqⁿ(1) $\Gamma(\frac{1}{2}) = \int_0^\infty x^{\frac{1}{2}-1} e^{-x}\, dx$

$\Rightarrow \Gamma(\frac{1}{2}) = \int_0^\infty x^{-\frac{1}{2}} e^{-x}\, dx$

put $x = y^2$
$\Rightarrow dx = 2y\,dy$
At $x=0,\ y=0$
At $x=\infty,\ y=\infty$

$= \int_0^\infty (y^2)^{-\frac{1}{2}} e^{-y^2} \cdot 2y\,dy$

$= 2\int_0^\infty y^{-1} \cdot e^{-y^2} \cdot y\,dy$

$= 2\int_0^\infty e^{-y^2}\, dy = 2\int_0^\infty e^{-x^2}\, dx$

$\therefore \Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-x^2}\, dx = 2\int_0^\infty e^{-y^2}\, dy$ —— (2)

Now $[\Gamma(\frac{1}{2})]^2 = 2\int_0^\infty e^{-x^2}\, dx \cdot 2\int_0^\infty e^{-y^2}\, dy$ using (2)

$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$

put $x = r\cos\theta,\ y = r\sin\theta$

$x^2+y^2 = (r^2\cos^2\theta + r^2\sin^2\theta)$
$= r^2(\cos^2\theta + \sin^2\theta)$
$= r^2 \cdot 1 = r^2$ & $\tan\theta = \frac{y}{x}$

When $y = 0 \Rightarrow \theta = 0$ & $y = \infty \Rightarrow \theta = \frac{\pi}{2}$

when $x = 0 \Rightarrow r = 0$, & $x = \infty \Rightarrow r = \infty$

$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$= r$

$$= 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2} |J|\, dr\, d\theta$$

$$= 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2} r\, dr\, d\theta$$

$$= 4\int_0^{\pi/2}\left(\int_0^{\infty} e^{-r^2} r\, dr\right) d\theta$$

$$= 4\int_0^{\pi/2}\left(\int_0^{\infty} e^{-t} \cdot \frac{1}{2} dt\right) d\theta$$

put $r^2 = t$
$2r\,dr = dt$
At $r = 0 \Rightarrow t = 0$
At $r = \infty \Rightarrow t = \infty$

$$= 2\int_0^{\pi/2} \left[-e^{-t}\right]_0^{\infty} d\theta$$

$$= 2\int_0^{\pi/2} \left[(0) - (-1)\right] d\theta$$

$$= 2\int_0^{\pi/2} d\theta = 2\left[\theta\right]_0^{\pi/2} = 2\left[\pi/2 - 0\right] = \pi$$

$$\therefore \left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = \pi$$

$$\Rightarrow \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \quad \text{——— (3)}$$

$\because \Gamma(n) > 0$ for $n > 0$

from (2) $\Gamma\left(\frac{1}{2}\right) = 2\int_0^{\infty} e^{-x^2} dx$

$$\Rightarrow \int_0^{\infty} e^{-x^2} dx = \frac{\Gamma\left(\frac{1}{2}\right)}{2} = \frac{\sqrt{\pi}}{2} \quad \text{——— (4)}$$

possible questions for (iv)

(1) Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ (do upto Eqn (3))

(2) Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and hence deduce that $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

(do upto eqn (4))

(3) Show that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ (do upto Eq(4))

(4) Show that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ (2 Marks)

Soln do upto Eqn(2) then

We know that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$\Rightarrow 2\int_0^\infty e^{-x^2} dx = \sqrt{\pi} \quad \text{from(2)}$$

$$\Rightarrow \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \quad \square$$

NOTE (i) Gamma function for -ve integers and 0 are undefined

for $\Gamma(n+1) = n\Gamma(n)$

$$\Rightarrow \Gamma(n) = \frac{\Gamma(n+1)}{n} \quad \text{--- (1)}$$

$$\Gamma(0) = \frac{\Gamma(0+1)}{0} = \frac{\Gamma(1)}{0} = \infty$$

$$\Gamma(-1) = \frac{\Gamma(0)}{-1} = \infty$$

(ii) $\Gamma(n+1) = n\Gamma(n)$ (Apply when n is +ve fraction)

$\Gamma(n+1) = \underline{|n}$ Apply when n is +ve integer.

$\Gamma(n) = \frac{\Gamma(n+1)}{n}$ Apply when n is -ve fraction.

① Compute $\Gamma(4)$

Soln $\Gamma(4) = \Gamma(3+1) = \lfloor 3 = 3\times2\times1 = 6$ (Ans)

② Compute $\Gamma(3.5)$

Soln $\Gamma(3.5) = \Gamma\left(\frac{35}{10}\right) = \Gamma\left(\frac{7}{2}\right)$

$= \Gamma\left(\frac{5}{2}+1\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right)$

$= \frac{5}{2}\cdot\Gamma\left(\frac{3}{2}+1\right) = \frac{5}{2}\cdot\frac{3}{2}\cdot\Gamma\left(\frac{3}{2}\right)$

$= \frac{15}{4}\cdot\Gamma\left(\frac{1}{2}+1\right) = \frac{15}{4}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)$

$= \frac{15}{8}\cdot\sqrt{\pi} = \frac{15\sqrt{\pi}}{8}$ (Ans)

③ Compute $\Gamma\left(-\frac{1}{2}\right)$

Soln; $\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{\left(-\frac{1}{2}\right)}$

$= -2\Gamma\left(\frac{1}{2}\right)$

$= -2\sqrt{\pi}$ (Ans)

④ Compute following:

(a) $\Gamma\left(\frac{3}{2}\right)$ (b) $\Gamma\left(\frac{5}{2}\right)$ (c) $\Gamma\left(\frac{7}{2}\right)$ (d) $\Gamma\left(-\frac{5}{2}\right)$

(e) $\frac{\Gamma\left(\frac{5}{2}\right)\cdot\Gamma\left(\frac{3}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma(3)}$ (f) $\Gamma(.5)$

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(g) $\frac{\Gamma(5/2)}{\Gamma(1/2)}$ (h) $\frac{\Gamma(7/3)}{\Gamma(4/3)}$ (i) $\Gamma(-9/2)$

(J) $\Gamma(-\frac{3}{2})$

LAPLACE Transform

(i) If $f(t)$ is a function of a real variable $t>0$ then Laplace transform of $f(t)$ is denoted by $L\{f(t)\}$ and defined as

$$\boxed{L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt}$$

where s is a parameter which may be real or complex.

(ii) since above integral after integration will be a function of 's' we may write

$$\boxed{L\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt}$$

piecewise contineous function:

A function $f(t)$ is said to be piecewise contineous in a finite range $a \leq t \leq b$ if it is possible to divide the range into finite number of sub intervals such that $f(t)$ is contineous in each sub interval and approaches finite values as t approaches either end of any interval from the interior.

Condition for existence of Laplace transform

Let $f(t)$ be a function which is piecewise contineous on every finite interval in the range $t \geq 0$ s.t $|f(t)| \leq Me^{\alpha t}$ for some constants α and M. then $L\{f(t)\}$ exist for all $s > \alpha$

LAPLACE TRANSFORM OF SOME STANDARD FUNCTION

(i) $L(K) = \frac{K}{s}$, $s>0$, $K=$ constant.

(ii) $L(t^n) = \frac{\lfloor n}{s^{n+1}}$, $s>0$, $n=0,1,2,3 \cdots$

(iii) $L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$, $s>0$ & n is a fraction, $n>-1$

(iv) $L\{e^{at}\} = \frac{1}{s-a}$, $s>a$

(v) $L\{\sin at\} = \frac{a}{s^2+a^2}$, $s>|a|$

(vi) $L\{\cos at\} = \frac{s}{s^2+a^2}$, $s>|a|$

(vii) $L\{\sinh at\} = \frac{a}{s^2-a^2}$, $s>|a|$

(viii) $L\{\cosh at\} = \frac{s}{s^2-a^2}$, $s>|a|$

PROOFS

(i) We know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt \quad \text{--- (1)}$$

using (1)

$$L\{K\} = \int_0^\infty e^{-st} \cdot K\,dt$$

$$= K\int_0^\infty e^{-st}\,dt$$

$$= K\left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= K\left[0 - \left(-\frac{1}{s}\right)\right]$$

$$= K \times \frac{1}{s} = \frac{K}{s} \quad \square$$

(ii) We know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt \quad \text{--- (1)}$$

using (1)

$$L\{t^n\} = \int_0^\infty e^{-st} \cdot t^n\,dt$$

$$= \left[t^n \int e^{-st}dt - \int\left(\int e^{-st}dt\right)\times\left(\frac{d}{dt}t^n\right)dt\right]_0^\infty$$

$$= \left[-\frac{t^n e^{-st}}{s} - \int \frac{e^{-st}}{-s} \times n t^{n-1}\,dt\right]_0^\infty$$

$$= \left[-\frac{t^n e^{-st}}{s}\right]_0^\infty + \frac{n}{s}\int_0^\infty e^{-st} \cdot t^{n-1}\,dt$$

$$[0-0] + \frac{n}{s}\int_0^\infty e^{-st} t^{n-1} dt$$

$$= \frac{n}{s}\int_0^\infty e^{-st} t^{n-1} dt \quad \text{——— (2)}$$

$$= \frac{n}{s}\left\{\left[t^{n-1}\int e^{-st}dt - \int\left(\int e^{-st}dt\right)\times\frac{d}{dt}(t^{n-1})\,dt\right]_0^\infty\right\}$$

$$= \frac{n}{s}\left\{\left[-\frac{t^{n-1}.e^{-st}}{s} - \int\frac{e^{-st}}{-s}\times(n-1)t^{n-2}dt\right]_0^\infty\right\}$$

$$= \frac{n}{s}\left\{\left[-\frac{t^{n-1}e^{-st}}{s}\right]_0^\infty + \frac{n-1}{s}\int_0^\infty e^{-st}t^{n-2}dt\right\}$$

$$= \frac{n}{s}\left\{(0-0) + \frac{(n-1)}{s}\int_0^\infty e^{-st}t^{n-2}dt\right\}$$

$$= \frac{n(n-1)}{s^2}\int_0^\infty e^{-st}t^{n-2}dt \quad \text{——— (3)}$$

Integrating n times we have

$$= \frac{n.(n-1)(n-2)------1}{s^n}\int_0^\infty e^{-st}t^{n-n}dt$$

$$= \frac{\lfloor n}{s^n}\times\int_0^\infty e^{-st}dt = \frac{\lfloor n}{s^n}\left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \frac{\lfloor n}{s^n}\times\left[0-\left(-\frac{1}{s}\right)\right] = \frac{\lfloor n}{s^n}\times\frac{1}{s} = \frac{\lfloor n}{s^{n+1}}$$

(iii) We know that $L\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt$ — (1)

using (1) $L\{t^n\} =$ ~~$\int_0^\infty e^{-st} f(t)\,dt$~~

$$= \int_0^\infty e^{-st}t^n\,dt$$

put $st = x$
$\Rightarrow s.dt = dx$
& $t = \frac{x}{s}$
At $t=0 \Rightarrow x=0$
At $t=\infty \Rightarrow x=\infty$

$$= \int_0^\infty e^{-x} \cdot \left(\frac{x}{s}\right)^n \cdot \frac{dx}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n \, dx$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^{(n+1)-1} \, dx$$

$$= \frac{1}{s^{n+1}} \Gamma(n+1) = \frac{\Gamma(n+1)}{s^{n+1}} \quad \square$$

(iv) We know that $L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$ —— (1)

using (1)

$$L\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at} \, dt$$

$$= \int_0^\infty e^{-(s-a)t} \, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= \left[0 - \left(-\frac{1}{s-a}\right)\right]$$

$\because s > a$
$\Rightarrow s-a > 0$
$-(s-a) < 0$

$$= \frac{1}{s-a} \quad \square$$

(V) We know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt \quad \text{——} (1)$$

using (1)

$$L\{\sin at\} = \int_0^\infty e^{-st} \sin at\, dt$$

$$= \left[\sin at \int e^{-st}\,dt - \int\left(\int e^{-st}\,dt\right)\left(\frac{d}{dt}\sin at\right)dt\right]_0^\infty$$

$$= \left[\frac{\sin at \cdot e^{-st}}{-s} - \int \frac{e^{-st}}{-s} \times a\cos at\,dt\right]_0^\infty$$

$$= \left[\frac{\sin at \cdot e^{-st}}{-s}\right]_0^\infty + \frac{a}{s}\int_0^\infty e^{-st}\cos at\,dt$$

$$= 0 + \frac{a}{s}\int_0^\infty e^{-st}\cos at\,dt$$

$$= \frac{a}{s}\left\{\left[\cos at \int e^{-st}\,dt - \int\left(\int e^{-st}\,dt\right)\left(\frac{d}{dt}\cos at\right)dt\right]_0^\infty\right\}$$

$$= \frac{a}{s}\left\{\left[\frac{\cos at \cdot e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} \times (-a\sin at)\,dt\right\}$$

$$= \frac{a}{s}\left\{0 - \left(-\frac{1}{s}\right) - \frac{a}{s}\int_0^\infty e^{-st}\sin at\,dt\right\}$$

$$= \frac{a}{s}\left\{\frac{1}{s} - \frac{a}{s} L(\sin at)\right\}$$

$$= \frac{a}{s^2} - \frac{a^2}{s^2} L\{\sin at\}$$

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$$\Rightarrow L(\sin at) + \frac{a^2}{s^2} L(\sin at) = \frac{a}{s^2}$$

$$\Rightarrow \left(1 + \frac{a^2}{s^2}\right) L(\sin at) = \frac{a}{s^2}$$

$$\Rightarrow \frac{s^2+a^2}{s^2} L(\sin at) = \frac{a}{s^2}$$

$$\Rightarrow L\{\sin at\} = \frac{a}{s^2} \times \frac{s^2}{(s^2+a^2)}$$

$$= \frac{a}{s^2+a^2}$$

$$\therefore L\{\sin at\} = \frac{a}{s^2+a^2} \quad \square$$

or

we know that

$$\sin at = \frac{e^{iat} - e^{-iat}}{2i}$$

$$\Rightarrow L\{\sin at\} = L\left\{\frac{e^{iat} - e^{-iat}}{2i}\right\}$$

$$= \frac{1}{2i} L\{e^{iat} - e^{-iat}\}$$

$$= \frac{1}{2i}\left[L\{e^{iat}\} - L\{e^{-iat}\}\right]$$

$$= \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right]$$

$$\frac{1}{2i}\left[\frac{(s+ia)-(s-ia)}{(s-ia)(s+ia)}\right]$$

$$=\frac{1}{2i}\left[\frac{s+ia-s+ia}{s^2-i^2a^2}\right]$$

$$=\frac{1}{2i}\left[\frac{2ia}{s^2-(-1)a^2}\right]=\frac{a}{s^2+a^2} \quad \square$$

(vi) $L\{\cos at)$ do like (V)

and Method এবং $\cos at = \frac{e^{iat}+e^{-iat}}{2}$

(vii) We know that

$$L\{f(t)\{ = \int_0^\infty e^{-st} f(t)\,dt \text{ ——— } (1)$$

using (1)

$$L\{\sinh at\{ = \int_0^\infty e^{-st} \sinh at\, dt$$

$$= \int_0^\infty e^{-st}\, \frac{e^{at}-e^{-at}}{2}\, dt$$

$$= \frac{1}{2}\int_0^\infty \left[e^{-(s-a)t} - e^{-(s+a)t}\right] dt$$

$$= \frac{1}{2}\left\{\int_0^\infty e^{-(s-a)t}dt - \int_0^\infty e^{-(s+a)t}dt\right\{$$

$$= \frac{1}{2}\left\{\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty - \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty\right\}$$

$$= \frac{1}{2}\left\{\left[0-\left(-\frac{1}{s-a}\right)\right]-\left[0-\left(-\frac{1}{s+a}\right)\right]\right\{$$

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$$= \frac{1}{2}\left\{ \frac{1}{s-a} - \frac{1}{s+a} \right\}$$

$$= \frac{1}{2}\left\{ \frac{(s+a)-(s-a)}{(s-a)(s+a)} \right\}$$

$$= \frac{1}{2}\left\{ \frac{s+a-s+a}{s^2-a^2} \right\} = \frac{1}{2} \times \frac{2a}{s^2-a^2} = \frac{a}{s^2-a^2} \quad \square$$

Or

We know that $\sinh at = \frac{e^{at}-e^{-at}}{2}$

$$\Rightarrow L\{\sinh at\} = L\left\{\frac{e^{at}-e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\left\{ L(e^{at}) - L(e^{-at}) \right\}$$

$$= \frac{1}{2}\left\{ \frac{1}{s-a} - \frac{1}{s+a} \right\} = \frac{1}{2}\left\{ \frac{(s+a)-(s-a)}{(s-a)(s+a)} \right\}$$

$$= \frac{1}{2}\left\{ \frac{s+a-s+a}{s^2-a^2} \right\} = \frac{1}{2}\left\{ \frac{2a}{s^2-a^2} \right\} = \frac{a}{s^2-a^2} \quad \square$$

(VIII) do like (VII), $\cosh at = \frac{e^{at}+e^{-at}}{2}$

LINEARITY PROPERTY

$$L\left\{\sum_{i=1}^{n} a_i f_i(t)\right\} = \sum_{i=1}^{n} a_i L\{f_i(t)\}$$

where a_i's are constants.

UNIT-STEP FUNCTION

(i) unit step function $u(t-a)$ is defined as

$$\boxed{u(t-a) = \begin{cases} 0, & t<a \\ 1, & t \geq a \end{cases}}$$

ii) $$\boxed{L\{u(t-a)\} = \frac{e^{-as}}{s}}$$

First shifting theorem:

If $L\{f(t)\} = \bar{f}(s)$ then $\boxed{L\{e^{at} f(t)\} = \bar{f}(s-a)}$

2nd shifting theorem:

If $L\{f(t)\} = \bar{f}(s)$ then $\boxed{L\{f(t-a).u(t-a)\} = e^{-as}.\bar{f}(s)}$

Multiplication by t^n

If $L\{f(t)\} = \bar{f}(s)$ then $\boxed{L\{t^n f(t)\} = (-1)^n \bar{f}^{(n)}(s)}$

Division by t:

If $L\{f(t)\} = \bar{f}(s)$ then $\boxed{L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\,ds}$

Transform of integral: If $L\{f(t)\} = \bar{f}(s)$ then

$$\boxed{L\left\{\int_0^t f(t)\,dt\right\} = \frac{\bar{f}(s)}{s}}$$

Transform of derivative: If $L\{f(t)\} = \bar{f}(s)$ then

$$\boxed{L\{f^{(n)}(t)\} = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)}$$

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(5) Find Laplace transform of $\sin(at+b)$

Soln $L\{\sin(at+b)\}$

$= L\{\sin at \cdot \cos b + \cos at \cdot \sin b\}$

$= L\{\sin at \cdot \cos b\} + L\{\cos at \cdot \sin b\}$

$= \cos b \cdot L\{\sin at\} + \sin b \cdot L\{\cos at\}$

$= \cos b \times \frac{a}{s^2+a^2} + \sin b \times \frac{s}{s^2+a^2}$

$= \frac{a\cos b}{s^2+a^2} + \frac{s \cdot \sin b}{s^2+a^2}$

$= \frac{a\cos b + s \cdot \sin b}{s^2+a^2}$ (Ans)

HOME TASK

(6) Find

(a) $L\{\cos(at+b)\}$

(b) $L\{\sin(at-b)\}$

(c) $L\{\cos(at-b)\}$

(d) $L\{\sin(2t+3)\}$

(e) $L\{\sin(t-1)\}$

(f) $L\{\cos(5t+3)\}$

(g) $L\{\cos(4t-3)\}$

(7) Find $L\{\sin^2 at\}$

Soln $L\{\sin^2 at\}$

$$= \frac{1}{2} L\{2\sin^2 at\}$$

$$= \frac{1}{2} L\{1 - \cos 2at\} \quad \left[\because 2\sin^2\theta = 1 - \cos 2\theta\right]$$

$$= \frac{1}{2} L(1) - \frac{1}{2} L(\cos 2at)$$

$$= \frac{1}{2} \times \frac{1}{s} - \frac{1}{2} \times \frac{s}{s^2 + (2a)^2}$$

$$= \frac{1}{2} \left\{ \frac{1}{s} - \frac{s}{s^2 + 4a^2} \right\}$$

$$= \frac{1}{2} \left\{ \frac{s^2 + 4a^2 - s^2}{s(s^2 + 4a^2)} \right\}$$

$$= \frac{1}{2} \times \frac{4a^2}{s(s^2 + 4a^2)}$$

$$= \frac{2a^2}{s(s^2 + 4a^2)} \quad \text{(Ans)}$$

HOME TASK

(8) Find Laplace transform of

(a) $\cos^2 at$

(b) $\sin^2 3t$

(c) $\cos^2 5t$.

(9) Find Laplace transform of $\sin^3 at$.

Soln $L\{\sin^3 at\}$

$$= L\left\{\frac{3\sin at - \sin 3at}{4}\right\}$$

$\because \sin 3A = 3\sin A - 4\sin^3 A$

$\Rightarrow \sin^3 A = \dfrac{3\sin A - \sin 3A}{4}$

$$= \frac{3}{4}L(\sin at) - \frac{1}{4}L(\sin 3at)$$

$$= \frac{3}{4} \times \frac{a}{s^2+a^2} - \frac{1}{4} \times \frac{3a}{s^2+(3a)^2}$$

$$= \frac{3a}{4}\left\{\frac{1}{s^2+a^2} - \frac{1}{s^2+9a^2}\right\}$$

$$= \frac{3a}{4}\left\{\frac{s^2+9a^2-(s^2+a^2)}{(s^2+a^2)(s^2+9a^2)}\right\}$$

$$\frac{3a}{4}\left\{\frac{s^2+9a^2-s^2-a^2}{(s^2+a^2)(s^2+9a^2)}\right\}$$

$$=\frac{3a}{4}\left\{\frac{8a^2}{(s^2+a^2)(s^2+9a^2)}\right\}$$

$$=\frac{6a^3}{(s^2+a^2)(s^2+9a^2)} \text{ (Ans)}$$

HOME TASK

(10) Find

(a) $L\{\cos^3 at\}$

(b) $L\{\sin^3 2t\}$

(c) $L\{\cos^3 5t\}$

Hints: $\cos 3A = 4\cos^3 A - 3\cos A$
$\Rightarrow \cos^3 A = \frac{\cos 3A + 3\cos A}{4}$

No 11 Find $L\{\cos at.\cos bt\}$

Soln $L\{\cos at.\cos bt\}$

$$=\frac{1}{2}L\{2\cos at.\cos bt\}$$

$$=\frac{1}{2}L\{\cos(at+bt)+\cos(at-bt)\}$$

$$=\frac{1}{2}L\{\cos(a+b)t+\cos(a-b)t\}$$

$$=\frac{1}{2}\left[L\{\cos(a+b)t\}+L\{\cos(a-b)t\}\right]$$

$$=\frac{1}{2}\left[\frac{s}{s^2+(a+b)^2}+\frac{s}{s^2+(a-b)^2}\right]$$

$$= \frac{s}{2}\left[\frac{1}{s^2+(a+b)^2} + \frac{1}{s^2+(a-b)^2}\right]$$

$$= \frac{s}{2}\left[\frac{s^2+(a-b)^2+s^2+(a+b)^2}{[s^2+(a+b)^2][s^2+(a-b)^2]}\right]$$

$$= \frac{s}{2}\left[\frac{2s^2+2(a^2+b^2)}{[s^2+(a+b)^2][s^2+(a-b)^2]}\right]$$

$$= \frac{s(s^2+a^2+b^2)}{[s^2+(a+b)^2][s^2+(a-b)^2]}$$ (Ans) ✓

HOME TASK

Q) Find

(a) $L(\sin at \sin bt)$

(b) $L(\sin at \cdot \cos bt)$

(c) $L(\sin 2t \cdot \sin 3t)$

(d) $L(\cos 2t \cdot \cos 5t)$

(e) $L(\sin 3t \cdot \cos 5t)$

(f) $L(\sin at \cdot \cos at)$

(g) $L(\sin 3t \cdot \cos 3t)$ ✓

(13) Find $L\{\sin t \cdot \sin 2t \cdot \sin 3t\}$

Soln $L\{\sin t \cdot \sin 2t \cdot \sin 3t\} = \frac{1}{2} L\{2\sin 2t \cdot \sin t \cdot \sin 3t\}$

$= \frac{1}{2} L\{[\cos(2t-t) - \cos(2t+t)]\sin 3t\}$

$= \frac{1}{2} L\{(\cos t - \cos 3t)\sin 3t\} = \frac{1}{2} L\{\cos t \sin 3t - \sin 3t \cdot \cos 3t\}$

$= \frac{1}{2} L\{\cos t \cdot \sin 3t\} - \frac{1}{2} L\{\sin 3t \cdot \cos 3t\}$

$= \frac{1}{4} L\{2\sin 3t \cdot \cos t\} - \frac{1}{4} L\{2\sin 3t \cdot \cos 3t\}$

$= \frac{1}{4} L\{\sin(3t+t) + \sin(3t-t)\} - \frac{1}{4} L\{\sin(2\times 3t)\}$

$= \frac{1}{4} L\{\sin 4t + \sin 2t\} - \frac{1}{4} L(\sin 6t)$

$= \frac{1}{4} L(\sin 4t) + \frac{1}{4} L(\sin 2t) - \frac{1}{4} L\{\sin 6t\}$

$= \frac{1}{4} \times \frac{4}{s^2+4^2} + \frac{1}{4} \times \frac{2}{s^2+2^2} - \frac{1}{4} \times \frac{6}{s^2+6^2}$

$= \frac{2}{4}\left\{\frac{2}{s^2+16} + \frac{1}{s^2+4} - \frac{3}{s^2+36}\right\}$

$= \frac{1}{2}\left\{\frac{1}{s^2+4} + \frac{2}{s^2+16} - \frac{3}{s^2+36}\right\}$ (Ans)

HOME TASK

(14) Find

(a) $L\{\cos t \cdot \cos 2t \cdot \cos 3t\}$

(b) $L\{\sin t \cdot \cos 2t \cdot \cos 3t\}$

(c) $L\{1 + 2t^3 - 4e^{3t} + 5e^{-t}\}$

(d) $L\{3\cosh 4t + 4\sin 3t\}$

(e) $L\{4\sinh 5t - 5\cos 4t\}$

(15) Find $L\{\sinh^2 at\}$.

Soln $L\{\sinh^2 at\} = L\left\{\frac{e^{at} - e^{-at}}{2}\right\}^2$

$= \frac{1}{4} L\{e^{at} - e^{-at}\}^2 = \frac{1}{4} L\{(e^{at})^2 + (e^{-at})^2 - 2 \cdot e^{at} \cdot e^{-at}\}$

$= \frac{1}{4} L\{e^{2at} + e^{-2at} - 2\}$ $\cancel{= \frac{1}{4}\left[L\{e^{2at}\} + L\{e^{-2at}\} - 2L\{1\}\right]}$

$\cancel{= \frac{1}{4}\left[\frac{1}{s-2a} + \frac{1}{s+2a} - \frac{2}{s}\right]}$

$= \frac{1}{2} L\left\{\frac{e^{2at} + e^{-2at}}{2} - 1\right\}$

$= \frac{1}{2} \cdot L\{\cosh 2at - 1\} = \frac{1}{2}\left[L\{\cosh 2at\} - L\{1\}\right]$

$= \frac{1}{2}\left[\frac{s}{s^2-(2a)^2} - \frac{1}{s}\right] = \frac{1}{2}\left[\frac{s}{s^2-4a^2} - \frac{1}{s}\right]$

$= \frac{1}{2}\left[\frac{s^2-(s^2-4a^2)}{s(s^2-4a^2)}\right] = \frac{1}{2}\left[\frac{s^2-s^2+4a^2}{s(s^2-4a^2)}\right]$

$= \frac{2a^2}{s(s^2-4a^2)}$ (Ans) ✓

HOME TASK

(16) Find

(a) $L\{\cosh^2 at\}$ (b) $L\{\sinh^2 5t\}$

(c) $L\{\cosh^2 3t\}$.

17. Prove that $L\left\{\frac{1}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}$

Soln $L\left\{\frac{1}{\sqrt{t}}\right\} = L\{t^{-\frac{1}{2}}\} = \frac{\Gamma(-\frac{1}{2}+1)}{s^{-\frac{1}{2}+1}}$

$= \frac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}$.

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HOME TASK

(18) Prove that

(a) $L\{\sqrt{t}\} = \dfrac{\sqrt{\pi}}{2s^{3/2}}$ (b) $L\{t^{3/2}\} = \dfrac{3\sqrt{\pi}}{4s^{5/2}}$

(c) $L\{t^{-3/2}\} = -2\sqrt{\pi s}$.

(19) Find

(a) $L\{(\sin t - \cos t)^2\}$

(b) $L\{t - \sinh at\}$

(20) Find $L\{\cosh at \cdot \sin at\}$.

Soln $L\{\cosh at \cdot \sin at\} = L\left\{\dfrac{e^{at}+e^{-at}}{2} \times \sin at\right\}$

$= \dfrac{1}{2} L\{e^{at}\sin at + e^{-at}\sin at\}$

$= \dfrac{1}{2}\{L(e^{at}\sin at) + L(e^{-at}\sin at)\}$

$= \dfrac{1}{2}\left\{\dfrac{a}{(s-a)^2+a^2} + \dfrac{a}{(s+a)^2+a^2}\right\}$

$= \dfrac{a}{2}\left\{\dfrac{1}{(s^2-2as+a^2+a^2)} + \dfrac{1}{s^2+2as+a^2+a^2}\right\}$

$= \dfrac{a}{2} \times \left\{\dfrac{1}{s^2-2as+2a^2} + \dfrac{1}{s^2+2as+2a^2}\right\}$

$= \dfrac{a}{2} \times \left\{\dfrac{s^2+2as+2a^2+s^2-2as+2a^2}{(s^2-2as+2a^2)(s^2+2as+2a^2)}\right\}$

$= \dfrac{a}{2} \times \dfrac{2s^2+4a^2}{(s^2+2a^2)^2-(2as)^2}$

$= \dfrac{a(s^2+2a^2)}{s^4+4a^4+4a^2s^2-4a^2s^2} = \dfrac{as^2+2a^3}{s^4+4a^4}$ (Ans)

Home TASK:

(21) Find

(a) $L(\sinh at . \cos at)$

(b) $L(\cosh 3t \sin 3t)$

(22) Find $L(te^{-t}\sin 4t)$

Soln $L(\sin 4t) = \frac{4}{s^2+4^2} = \frac{4}{s^2+16} = \bar{f}(s)$ (say)

where $f(t) = \sin 4t$

Now $\bar{f}'(s) = \frac{d}{ds}\bar{f}(s) = \frac{d}{ds}\left\{\frac{4}{(s^2+16)}\right\}$

$= \cancel{4 \times \frac{d}{ds}(s^2+16) - (s^2+16)\frac{d}{ds}(4)}$

$= \frac{(s^2+16)\frac{d}{ds}(4) - (4)\frac{d}{ds}(s^2+16)}{(s^2+16)^2}$

$= \frac{(s^2+16)\times 0 - 4\times 2s}{(s^2+16)^2} = \frac{-8s}{(s^2+16)^2}$ ✓

Now $L(t.\sin 4t)$

$= L(t.f(t)) = (-1)^1 \bar{f}'(s)$

$= -\bar{f}'(s) = \frac{8s}{(s^2+16)^2} = \bar{g}(s)$ (say)

where $g(t) = t.\sin 4t$

Now $L(t.e^{-t}.\sin 4t)$

$= L\{e^{-t}(t\sin 4t)\}$

$$= L\{e^{-t} \cdot g(t)\} = \bar{g}(s+1)$$

$$= \frac{8(s+1)}{[(s+1)^2+16]^2} = \frac{8(s+1)}{[s^2+2s+1+16]^2} = \frac{8(s+1)}{(s^2+2s+17)^2}$$

HOME TASK

(23) Find

(a) $L(e^{2t} \cdot t^5)$

(b) $L(e^{3t} \cdot \sin^2 t)$

(c) $L\{e^{-t}(3\sinh 2t - 2\cosh 3t)\}$

(d) $L\{e^{-4t} \cos t \cdot \sin 2t\}$

(e) $L\{e^{-t} \sin 4t + t\cos 2t\}$

(f) $L\{t \sin^2 t\}$

(g) $L\{t^2 \cos at\}$

(h) $L\{t \sin 3t \cdot \cos 2t\}$

(i) $L\{t e^{-2t} \sin 2t\}$

(j) $L\{t e^{-t} \cosh t\}$

(k) $L\{t^2 e^{-2t} \cdot \cos t\}$

(24) Find $L\left\{\dfrac{e^{-at} - e^{-bt}}{t}\right\}$

Solution $L\{e^{-at} - e^{-bt}\}$

$$= L\{e^{-at}\} - L\{e^{-bt}\}$$

$$= \frac{1}{s+a} - \frac{1}{s+b} = F(s) \text{ (say)}$$

Where $f(t) = e^{-at} - e^{-bt}$

Now $L\left\{ \frac{e^{-at} - e^{-bt}}{t} \right\} = L\left\{ \frac{f(t)}{t} \right\} = \int_s^\infty \bar{f}(s)\,ds$

$$= \int_s^\infty \left[\frac{1}{s+a} - \frac{1}{s+b}\right] ds = \int_s^\infty \frac{ds}{s+a} - \int_s^\infty \frac{1}{s+b}\,ds$$

$$= \left[\log(s+a)\right]_s^\infty - \left[\log(s+b)\right]_s^\infty$$

$$= \left[\log(s+a) - \log(s+b)\right]_s^\infty$$

$$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty = \left[\log\left(\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right)\right]_s^\infty$$

$$= \left[\log 1 - \log\left(\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right)\right]$$

$$= 0 - \log\left(\frac{s+a}{s+b}\right) = \log\left(\frac{s+a}{s+b}\right)^{-1}$$

$$= \log\left(\frac{s+b}{s+a}\right) \quad \text{(Ans)}$$

HOME TASK

(25) find

imp (a) $L\left\{ \frac{\cos 2t - \cos 3t}{t} \right\}$

(b) $L\left\{ \frac{e^{-t}\sin t}{t} \right\}$

(c) $L\left\{ \frac{1-\cos t}{t^2} \right\}$

(d) $L\left\{ \frac{\sin at}{t} \right\}$

imp (e) $L\left\{ \frac{\cos at - \cos bt}{t} \right\}$

(26) Find ~~L{e^{-3t} sin t}~~ $L\left\{\int_0^t e^t \cdot \left(\frac{\sin t}{t}\right) dt\right\}$

Solution:

$L\{\sin t\} = \frac{1}{s^2+1^2} = \frac{1}{s^2+1} = \bar{f}(s)$ (say)

where $f(t) = \sin t$.

Now $L\left\{\frac{\sin t}{t}\right\} = L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\, ds$

$= \int_s^\infty \left(\frac{1}{s^2+1}\right) ds = \left[\tan^{-1} s\right]_s^\infty = \tan^{-1}\infty - \tan^{-1} s$

$= \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s = \bar{g}(s)$ say

where $g(t) = \frac{\sin t}{t}$.

Now $L\left\{e^t \cdot \left(\frac{\sin t}{t}\right)\right\} = L\{e^t \cdot g(t)\} = \bar{g}(s-1)$

$= \cot^{-1}(s-1) = \bar{h}(s)$ (say) where

$h(t) = e^t\left(\frac{\sin t}{t}\right)$

Now $L\left\{\int_0^t e^t \cdot \left(\frac{\sin t}{t}\right) dt\right\} = L\left\{\int_0^t h(t)\, dt\right\}$

$= \frac{\bar{h}(s)}{s} = \frac{\cot^{-1}(s-1)}{s}$ (Ans)

HOME TASK

(27) Find (a) $L\left\{\int_0^t e^{-t} \cos t\, dt\right\}$

(b) $L\left\{\int_0^t \frac{\sin t}{t}\, dt\right\}$

(c) $L\left\{\int_0^t \frac{\cos t}{t}\, dt\right\}$.

(28) Using Laplace transform find

$$\int_0^\infty t e^{-3t} \sin t \, dt$$

Solution

We know that $L\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t) dt$ —(1)

Let $f(t) = t \sin t$ & $s = 3$

$L\{\sin t\} = \frac{1}{s^2+1^2} = \frac{1}{s^2+1} = \bar{g}(s)$ (say)

where $g(t) = \sin t$

$\bar{g}'(s) = \frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{(s^2+1)\frac{d}{ds}(1) - (1)\frac{d}{ds}(s^2+1)}{(s^2+1)^2}$

$= \frac{-2s}{(s^2+1)^2}$

Now $L\{f(t)\} = L\{t \sin t\} = L\{t^1 \cdot g(t)\} = (-1)^1 \bar{g}'(s)$

$= \frac{2s}{(s^2+1)^2}$

Now $\int_0^\infty t e^{-3t} \sin t \, dt = \int_0^\infty e^{-3t} (t \sin t) dt$

$= \int_0^\infty e^{-st} f(t) dt = L\{f(t)\} = \frac{2s}{(s^2+1)^2}$

$= \frac{2 \times 3}{(3^2+1)^2} = \frac{6}{100} = \frac{3}{50}$ (Ans)

HOME TASK

(29) Using Laplace transform find

(a) $\int_0^\infty t e^{-2t} \cos t \, dt$ (b) $\int_0^\infty e^{-3t} t^3 \cos t \, dt$.

(30) Using Laplace transform of $\cos at$, find Laplace transform of $\sin at$.

Soln We know that $L\{\cos at\} = \frac{s}{s^2+a^2} = \bar{f}(s)$

(say) where $f(t) = \cos at$.

Now $f'(t) = \frac{d}{dt}(\cos at) = -a \sin at$.

$\Rightarrow \cancel{L\{f'(t)\} = L\{a \sin t\} = aL\{\sin}$

$\Rightarrow L\{f'(t)\} = L\{-a \sin at\}$

$\Rightarrow s\bar{f}(s) - f(0) = -aL\{\sin at\}$

$\Rightarrow s \times \frac{s}{s^2+a^2} - 1 = -aL\{\sin at\}$

$\Rightarrow \frac{s^2}{s^2+a^2} - 1 = -aL\{\sin at\}$

$\Rightarrow \frac{s^2 - s^2 - a^2}{s^2+a^2} = -aL\{\sin at\}$

$\Rightarrow L\{\sin at\} = \frac{a}{s^2+a^2}$ (Ans)

HOME TASK

(31) Using Laplace transform of $\sin at$, find Laplace transform of $\cos at$.

INVERSE LAPLACE Transform:

(i) If $f(t)$ is a function of real variable $t > 0$ then Laplace transform of $f(t)$ is denoted by $L\{f(t)\}$ and defined as

$$L\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt \quad \text{—— (1)}$$

Where s is a parameter which may be real or complex.

(ii) If $L\{f(t)\} = \bar{f}(s)$ then $f(t)$ is called inverse Laplace transform of $\bar{f}(s)$ which is denoted as $L^{-1}\{\bar{f}(s)\}$.

ie If $L\{f(t)\} = \bar{f}(s) \Leftrightarrow L^{-1}\{\bar{f}(s)\} = f(t)$.

SOME STANDARD FORMULA

(1) $L^{-1}\left(\frac{K}{s}\right) = K$, $K =$ constant.

(2) $L^{-1}\left\{\frac{\lfloor n}{s^{n+1}}\right\} = t^n$, $n = 0, 1, 2, 3, \ldots$

(3) $L^{-1}\left\{\frac{\Gamma(n+1)}{s^{n+1}}\right\} = t^n$, n is a fraction.

(4) $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

(5) $L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$

(6) $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

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7) $L^{-1}\left(\frac{a}{s^2-a^2}\right) = \sinh at$

8) $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at.$

9) $L^{-1}\left(\frac{e^{-as}}{s}\right) = u(t-a)$ ✓

10) $L^{-1}\left(\bar{f}(s-a)\right) = e^{at}\cdot f(t).$

11) $L^{-1}\{e^{-as}\cdot\bar{f}(s)\} = f(t-a)\cdot u(t-a)$ (Heaviside shift thm)

12) $L^{-1}\{\bar{f}^{(n)}(s)\} = (-1)^n\cdot t^n\cdot f(t)$

13) $L^{-1}\{\int_s^\infty \bar{f}(s)\,ds\} = \frac{f(t)}{t}$

14) $L^{-1}\{\frac{\bar{f}(s)}{s}\} = \int_0^t f(t)\,dt.$

15) $L^{-1}\{\frac{a}{(s-b)^2+a^2}\} = e^{bt}\cdot \sin at.$

16) $L^{-1}\{\frac{s-b}{(s-b)^2+a^2}\} = e^{bt}\cdot\cos at$

(17) $L^{-1}\{\frac{a}{(s-b)^2-a^2}\} = e^{bt}\sinh at$

18) $L^{-1}\{\frac{s-b}{(s-b)^2-a^2}\} = e^{bt}\cosh at$

using No(10).

(19) $L^{-1}\{\frac{s}{(s^2+a^2)^2}\} = \frac{1}{2a}t\sin at$

(20) $L^{-1}\{\frac{1}{(s^2+a^2)^2}\} = \frac{1}{2a^3}\{\sin at - at\cos at\}$

(21) $L^{-1}\{\frac{s}{(s^2-a^2)^2}\} = \frac{1}{2a}t\sinh at.$

(22) $L^{-1}\{\frac{1}{(s^2-a^2)^2}\} = -\frac{1}{2a^3}[\sinh at - at\cosh at]$

Methods of Evaluation of Inverse LT

(i) using standard formula
(ii) using partial fraction

32) Find Inverse Laplace transform of

$$\frac{3s-4}{16-s^2}$$

Soln $L^{-1}\left\{\frac{3s-4}{16-s^2}\right\}$

$= L^{-1}\left\{\frac{4-3s}{s^2-16}\right\} = L^{-1}\left\{\frac{4}{s^2-16} - \frac{3s}{s^2-16}\right\}$

$= L^{-1}\left\{\frac{4}{s^2-16}\right\} - 3L^{-1}\left\{\frac{s}{s^2-16}\right\}$

$= L^{-1}\left\{\frac{4}{s^2-4^2}\right\} - 3L^{-1}\left\{\frac{s}{s^2-(4)^2}\right\}$

$= \sinh 4t - 3\cosh 4t$ (Ans)

33) Find $L^{-1}\left\{\frac{s+2}{s^2-4s+13}\right\}$

Soln $L^{-1}\left\{\frac{s+2}{s^2-4s+13}\right\} = L^{-1}\left\{\frac{s+2}{(s)^2-2(s)(2)+2^2-2^2+13}\right\}$

$= L^{-1}\left\{\frac{s+2}{(s-2)^2+9}\right\} = L^{-1}\left\{\frac{(s-2)+4}{(s-2)^2+3^2}\right\}$

$= L^{-1}\left\{\frac{s-2}{(s-2)^2+3^2}\right\} + L^{-1}\left\{\frac{4}{(s-2)^2+3^2}\right\}$

$= L^{-1}\left\{\frac{s-2}{(s-2)^2+3^2}\right\} + \frac{4}{3}L^{-1}\left\{\frac{3}{(s-2)^2+3^2}\right\}$

$= e^{2t}\cos 3t + \frac{4}{3} e^{2t}\sin 3t$

$= e^{2t}(\cos 3t + \frac{4}{3}\sin 3t)$ (Ans)

34) find $L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\}$

Soln Let $\bar{f}(s) = \frac{1}{s^2+a^2}$ ——— ①

$\Rightarrow L\{f(t)\} = \frac{1}{s^2+a^2}$

$\Rightarrow f(t) = L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a} L^{-1}\left\{\frac{a}{s^2+a^2}\right\}$

$\Rightarrow f(t) = \frac{1}{a}\sin at$ ——— ②

Now $L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+a^2}\right\}$

$= L^{-1}\left\{\frac{1}{s}\cdot \bar{f}(s)\right\} = L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)\,dt$

$= \int_0^t \frac{1}{a}\sin at\,dt = \frac{1}{a}\int_0^t \sin at\,dt$

$= \frac{1}{a}\left[-\frac{\cos at}{a}\right]_0^t = -\frac{1}{a^2}\left[\cos at\right]_0^t =$

$= -\frac{1}{a^2}\left[\cos at - 1\right]$ (Ans)

No 35 find $L^{-1}\left\{\log\left(\frac{s+a}{s+b}\right)\right\}$

Soln Let $\bar{f}(s) = \log\left(\frac{s+a}{s+b}\right)$ ——— ①

$\Rightarrow L\{f(t)\} = \log\left(\frac{s+a}{s+b}\right)$

$\Rightarrow L^{-1}\left\{\log\left(\frac{s+a}{s+b}\right)\right\} = f(t)$ ——— ②

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$$\bar{f}'(s) = \frac{d}{ds}\bar{f}(s) = \frac{d}{ds}\log\left(\frac{s+a}{s+b}\right)$$

$$= \frac{d}{ds}\left[\log(s+a) - \log(s+b)\right]$$

$$= \frac{d}{ds}\log(s+a) - \frac{d}{ds}\log(s+b)$$

$$= \frac{1}{s+a}\cdot\frac{d}{ds}(s+a) - \frac{1}{s+b}\cdot\frac{d}{ds}(s+b)$$

$$= \frac{1}{s+a}\times 1 - \frac{1}{s+b}\times 1 = \frac{1}{s+a} - \frac{1}{s+b}$$

$$\Rightarrow L^{-1}\{\bar{f}'(s)\} = L^{-1}\left\{\frac{1}{s+a} - \frac{1}{s+b}\right\}$$

$$\Rightarrow (-1)^1 t^1 f(t) = L^{-1}\left(\frac{1}{s+a}\right) - L^{-1}\left(\frac{1}{s+b}\right)$$

$$\Rightarrow -t\cdot f(t) = e^{-at} - e^{-bt}$$

$$\Rightarrow f(t) = -\frac{1}{t}\left(e^{-at} - e^{-bt}\right)$$

$$\Rightarrow L^{-1}\left\{\log\left(\frac{s+a}{s+b}\right)\right\} = -\frac{1}{t}\left(e^{-at} - e^{-bt}\right)$$

No 36 Find $L^{-1}\left\{\dfrac{s^2+s-2}{s(s+3)(s-2)}\right\}$

Solⁿ: Let $\dfrac{s^2+s-2}{s(s+3)(s-2)} = \dfrac{A}{s} + \dfrac{B}{s+3} + \dfrac{C}{s-2}$ ——(1)

$$\Rightarrow s^2+s-2 = A(s+3)(s-2) + B\,s(s-2) + C\,s(s+3)$$ ——(2)

put $s=0$ in Eqⁿ(2)

$\Rightarrow -2 = -6A \Rightarrow A = \frac{2}{6} = \frac{1}{3}$

put $s=2$ in eqⁿ(2)

$\Rightarrow 4 = 10C \Rightarrow C = \frac{4}{10} = \frac{2}{5}$

put $s=-3$ in equation (2)

$\Rightarrow 4 = 15B \Rightarrow B = \frac{4}{15}$

Now from eqⁿ(1)

$$\frac{s^2+s-2}{s(s+3)(s-2)} = \frac{1/3}{s} + \frac{4/15}{s+3} + \frac{2/5}{s-2} \quad \text{--- (3)}$$

Now $L^{-1}\left\{\frac{s^2+s-2}{s(s+3)(s-2)}\right\} = L^{-1}\left\{\frac{1/3}{s} + \frac{4/15}{s+3} + \frac{2/5}{s-2}\right\}$

$= \frac{1}{3}L^{-1}\left\{\frac{1}{s}\right\} + \frac{4}{15}L^{-1}\left\{\frac{1}{s+3}\right\} + \frac{2}{5}L^{-1}\left\{\frac{1}{s-2}\right\}$

$= \frac{1}{3}\times 1 + \frac{4}{15}e^{-3t} + \frac{2}{5}e^{2t}$

$= \frac{1}{3} + \frac{4}{15}e^{-3t} + \frac{2}{5}e^{2t}$ ✓

37) Find $L^{-1}\left\{\frac{s^2-10s+13}{(s-7)(s^2-5s+6)}\right\}$

Solⁿ Let $\frac{s^2-10s+13}{(s-7)(s^2-5s+6)} = \frac{s^2-10s+13}{(s-7)(s^2-2s-3s+6)}$

$= \frac{s^2-10s+13}{(s-7)\{s(s-2)-3(s-2)\}} = \frac{s^2-10s+13}{(s-7)(s-2)(s-3)}$

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$$= \frac{A}{s-7} + \frac{B}{s-2} + \frac{C}{s-3}$$

$$\Rightarrow \frac{s^2-10s+13}{(s-7)(s^2-5s+6)} = \frac{A}{(s-7)} + \frac{B}{s-2} + \frac{C}{s-3} \quad \text{—— (1)}$$

$$\Rightarrow \cancel{\frac{s^2-10s+13}{(s-7)(s^2-5s+6)}}$$

$$\Rightarrow s^2-10s+13 = A(s-2)(s-3) + B(s-7)(s-3) + C(s-7)(s-2) \quad \text{—— (2)}$$

put $s=2$ in Eqn(2),

$$\Rightarrow -3 = 5B \Rightarrow \boxed{B = -3/5}$$

put $s=3$ in Eqn(2)

$$\Rightarrow -8 = -4C \Rightarrow \boxed{C = 2}$$

put $s=7$ in Eqn(2)

$$\Rightarrow -8 = 20A \Rightarrow \boxed{A = -\frac{8}{20} = -\frac{2}{5}}$$

Now from (1)

$$\frac{s^2-10s+13}{(s-7)(s^2-5s+6)} = \frac{(-2/5)}{(s-7)} + \frac{(-3/5)}{(s-2)} + \frac{2}{s-3} \quad \text{—— (3)}$$

$$\text{Now } L^{-1}\left\{\frac{s^2-10s+13}{(s-7)(s^2-5s+6)}\right\}$$

$$= L^{-1}\left\{\frac{(-2/5)}{s-7} + \frac{(-3/5)}{s-2} + \frac{2}{s-3}\right\}$$

$$= -\frac{2}{5}L^{-1}\left\{\frac{1}{s-7}\right\} - \frac{3}{5}L^{-1}\left\{\frac{1}{s-2}\right\} + 2L^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= -\frac{2}{5}e^{7t} - \frac{3}{5}e^{2t} + 2e^{3t} \quad \text{(Ans)}$$

No 38 Find $L^{-1}\left\{\frac{1+2s}{(s+2)^2(s-1)^2}\right\}$

Sol^n Let $\frac{1+2s}{(s+2)^2(s-1)^2} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2}$ —— (1)

$\Rightarrow 1+2s = A(s+2)(s-1)^2 + B(s-1)^2 + C(s+2)^2(s-1) + D(s+2)^2$ —— (2)

put $s=1$ in eq^n (2)

$\Rightarrow 3 = 9D \Rightarrow D = \frac{3}{9} \Rightarrow \boxed{D = \frac{1}{3}}$

put $s=-2$ in Eq^n (2) $\Rightarrow -3 = 9B \Rightarrow \boxed{B = \frac{-3}{9} = -\frac{1}{3}}$

Now Equating co-efficients of s^2 in both sides of (2)

$B + 3C + D = 0 \Rightarrow -\frac{1}{3} + 3C + \frac{1}{3} = 0$

$\Rightarrow 3C = 0 \Rightarrow \boxed{C = 0}$

Now Equating co-efficients of s^3 in both sides of (2)

$A + C = 0 \Rightarrow \boxed{A = -C = 0}$

Now from Equation (1), Sunday 14

$$\frac{1+2s}{(s+2)^2(s-1)^2} = \frac{(-1/3)}{(s+2)^2} + \frac{(1/3)}{(s-1)^2} \quad \text{—— (3)}$$

Now $L^{-1}\left\{\frac{1+2s}{(s+2)^2(s-1)^2}\right\} = L^{-1}\left\{\frac{(-1/3)}{(s+2)^2} + \frac{(1/3)}{(s-1)^2}\right\}$

$$= -\frac{1}{3}L^{-1}\left\{\frac{1}{(s+2)^2}\right\} + \frac{1}{3}L^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$= -\frac{1}{3}L^{-1}\left\{\frac{\underline{1}}{(s+2)^{1+1}}\right\} + \frac{1}{3}L^{-1}\left\{\frac{\underline{1}}{(s-1)^{1+1}}\right\}$$

$$= -\frac{1}{3}e^{-2t}.t + \frac{1}{3}e^{t}.t$$

$$= \frac{1}{3}t\left(e^{t} - e^{-2t}\right) \text{ (Ans).}$$

No 39) Find $L^{-1}\left\{\frac{s}{(s-3)(s^2+4)}\right\}$

Soln Let $\frac{s}{(s-3)(s^2+4)} = \frac{A}{s-3} + \frac{Bs+C}{s^2+4}$ —— (1)

$\Rightarrow s = A(s^2+4) + (Bs+C)(s-3)$ —— (2)

put $s=3$ in equation (2)

$\Rightarrow 3 = 13A \Rightarrow \boxed{A = \frac{3}{13}}$

Now Equating constant terms in Eqn (2)

$$4A - 3C = 0$$

$$\Rightarrow 4 \times \frac{3}{13} - 3C = 0 \Rightarrow 3C = \frac{12}{13}$$

$$\Rightarrow \boxed{C = \frac{4}{13}}$$

Equating co-efficients of s^2 in Eqn (2)

$$A + B = 0$$

$$\Rightarrow \boxed{B = -A = -\frac{3}{13}}$$

Now from ①

$$\frac{s}{(s-3)(s^2+4)} = \frac{(3/13)}{s-3} + \frac{-\frac{3}{13}s+\frac{4}{13}}{s^2+4}$$

$$\Rightarrow \frac{s}{(s-3)(s^2+4)} = \frac{(3/13)}{(s-3)} - \frac{\frac{3}{13}s}{s^2+4} + \frac{4/13}{s^2+4}$$

$$\Rightarrow L^{-1}\left\{\frac{s}{(s-3)(s^2+4)}\right\} = \frac{3}{13}L^{-1}\left\{\frac{1}{s-3}\right\} - \frac{3}{13}L^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{4}{13}L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= \frac{3}{13}e^{3t} - \frac{3}{13}L^{-1}\left\{\frac{s}{s^2+2^2}\right\} + \frac{2}{13}L^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= \frac{3}{13}e^{3t} - \frac{3}{13}\cos 2t + \frac{2}{13}\sin 2t$$

$$= \frac{1}{13}\left\{3e^{3t} - 3\cos 2t + 2\sin 2t\right\}$$ (Ans)

No 40)) find $L^{-1}\left\{\frac{e^{-3s}}{(s-4)^2}\right\}$

Soln Let $\bar{f}(s) = \frac{1}{(s-4)^2}$ ——①

$$\Rightarrow L\{f(t)\} = \frac{1}{(s-4)^2} \Rightarrow f(t) = L^{-1}\left\{\frac{1}{(s-4)^2}\right\}$$

$$\Rightarrow f(t) = L^{-1}\left\{\frac{1!}{(s-4)^{1+1}}\right\} = t\cdot e^{4t}$$ ——②

Now $L^{-1}\left\{\frac{e^{-3s}}{(s-4)^2}\right\} = L^{-1}\left\{e^{-3s}\cdot\frac{1}{(s-4)^2}\right\}$

$$= L^{-1}\left\{e^{-3s}\,\bar{f}(s)\right\} = f(t-3)\,u(t-3)$$

$$= (t-3)e^{4(t-3)}\,u(t-3)$$ (Ans)

~~= 0 when t<3~~

$= (t-3)e^{4(t-3)}$ when $t \geq 3$

No 41)) find $L^{-1}\left\{\frac{e^{-3s}}{s}\right\}$

soln $L^{-1}\left\{\frac{e^{-3s}}{s}\right\}$

$= u(t-3)$ (Ans) $\because L^{-1}\left\{\frac{e^{-as}}{s}\right\} = u(t-a)$

$= 0$ when $t<3$

$\ 1$ when $t \geq 3$.

42)) find $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

soln Let $\bar{f}(s) = \frac{1}{(s^2+a^2)}$ —— (1)

$\Rightarrow L\{f(t)\} = \frac{1}{s^2+a^2} \Rightarrow f(t) = L^{-1}\left\{\frac{1}{s^2+a^2}\right\}$

$\Rightarrow f(t) = \frac{1}{a} \cdot L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \frac{1}{a}\sin at$ —— (2)

Now $\bar{f}'(s) = \frac{d}{ds}\bar{f}(s) = \frac{d}{ds}\left\{\frac{1}{(s^2+a^2)}\right\}$

$$= \frac{(s^2+a^2)\frac{d}{ds}(1) - (1)\frac{d}{ds}(s^2+a^2)}{(s^2+a^2)^2}$$

$$= \frac{-2s}{(s^2+a^2)^2}$$

$\therefore \bar{f}'(s) = \frac{-2s}{(s^2+a^2)^2}$

$\Rightarrow L^{-1}\{\bar{f}'(s)\} = L^{-1}\left\{\frac{-2s}{(s^2+a^2)^2}\right\}$

$\Rightarrow (-1)^1 t^1 f(t) = -2 \cdot L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

$\Rightarrow -t \cdot f(t) = -2 \cdot L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

$\Rightarrow L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2} t \cdot f(t)$

$= \frac{1}{2} t \cdot \frac{1}{a} \sin at$

$= \frac{1}{2a} t \sin at$ (Ans)

No 43 Find $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$

Soln Let $\bar{f}(s) = \frac{s}{(s^2+a^2)^2}$ —— (1)

$\Rightarrow L\{f(t)\} = \frac{s}{(s^2+a^2)^2}$

$\Rightarrow f(t) = L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a} t \sin at$ —— (2)

Now $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{s}{(s^2+a^2)^2} \cdot \frac{1}{s}\right\}$

$= L^{-1}\left\{\bar{f}(s) \cdot \frac{1}{s}\right\} = L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)\,dt$

$= \int_0^t \frac{1}{2a} t \sin at\,dt = \frac{1}{2a}\int_0^t t \sin at \cdot dt$

$= \frac{1}{2a}\left[t\int \sin at\,dt - \int\left(\int \sin at \cdot dt\right) \times \left(\frac{d}{dt} t\right)dt\right]_0^t$

$= \frac{1}{2a}\left[\frac{-t\cos at}{a} - \int -\frac{\cos at}{a} \times 1 \times dt\right]_0^t$

$= \frac{1}{2a}\left[\frac{-t\cos at}{a} + \frac{1}{a}\int \cos at\,dt\right]_0^t$

$= \frac{1}{2a}\left[-\frac{t\cos at}{a} + \frac{1}{a} \times \frac{\sin at}{a}\right]_0^t$

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$$= \frac{1}{2a}\left[-\frac{t\cos at}{a} + \frac{\sin at}{a^2}\right]_0^t$$

$$= \frac{1}{2a}\left[\left(-\frac{t\cos at}{a} + \frac{\sin at}{a^2}\right) - 0\right]$$

$$= \frac{1}{2a}\left[\frac{-at\cos at + \sin at}{a^2}\right]$$

$$= \frac{1}{2a^3}\left[\sin at - at\cos at\right] \quad \text{(Ans)}$$

44) Find $L^{-1}\left\{\frac{s}{(s^2-a^2)^2}\right\}$

Soln Let $\bar{f}(s) = \frac{1}{(s^2-a^2)}$ —— (1)

$\Rightarrow L\{f(t)\} = \frac{1}{(s^2-a^2)} \Rightarrow f(t) = L^{-1}\left\{\frac{1}{(s^2-a^2)}\right\}$

$\Rightarrow f(t) = \frac{1}{a} \cdot L^{-1}\left\{\frac{a}{(s^2-a^2)}\right\}$

$\Rightarrow f(t) = \frac{1}{a}\sinh(at)$ —— (2)

Now $\bar{f}'(s) = \frac{d}{ds}\frac{1}{(s^2-a^2)} = \frac{-2s}{(s^2-a^2)^2}$

$\Rightarrow L^{-1}\{\bar{f}'(s)\} = L^{-1}\left\{\frac{-2s}{(s^2-a^2)^2}\right\}$

$\Rightarrow (-1)^1 t^1 f(t) = -2\, L^{-1}\left\{\frac{s}{(s^2-a^2)^2}\right\}$

$\Rightarrow -t \cdot f(t) = -2 \cdot L^{-1}\left\{\frac{s}{(s^2-a^2)^2}\right\}$

$\Rightarrow L^{-1}\left\{\frac{s}{(s^2-a^2)^2}\right\} = \frac{1}{2}t\cdot f(t) = \frac{1}{2}t\cdot\frac{1}{a}\sinh at$

$= \frac{1}{2a}t\sinh at$

(45) find $L^{-1}\left\{\frac{1}{(s^2-a^2)^2}\right\}$

Soln Let $\bar{f}(s) = \frac{s}{(s^2-a^2)^2}$ —— ①

$\Rightarrow L\{f(t)\} = \frac{s}{(s^2-a^2)^2} \Rightarrow f(t) = L^{-1}\left\{\frac{s}{(s^2-a^2)^2}\right\}$

$\Rightarrow f(t) = \frac{1}{2a}t\sinh at$ —— ②

Now $L^{-1}\left\{\frac{1}{(s^2-a^2)^2}\right\} = L^{-1}\left\{\frac{s}{(s^2-a^2)^2}\cdot\frac{1}{s}\right\}$

$= L^{-1}\left\{\bar{f}(s)\cdot\frac{1}{s}\right\} = L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)\,dt$

$= \int_0^t \frac{1}{2a}t\sinh at\,dt$

$= \frac{1}{2a}\int_0^t t\sinh at\,dt$

$= \frac{1}{2a}\left[t\int\sinh at\,dt - \int\left(\int\sinh at\,dt\right)\times\left(\frac{d}{dt}t\right)\times dt\right]_0^t$

$= \frac{1}{2a}\left[\frac{t\cosh at}{a} - \int\frac{\cosh at}{a}\times1\times dt\right]_0^t$

$= \frac{1}{2a}\left[\frac{t\cosh at}{a} - \frac{\sinh at}{a^2}\right]_0^t$

$= \frac{1}{2a}\left[\left(\frac{t\cosh at}{a} - \frac{\sinh at}{a^2}\right) - 0\right]$

$= \frac{1}{2a}\left[\frac{at\cosh at - \sinh at}{a^2}\right]$

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$$\frac{1}{2a^3}\left[at\cosh at - \sinh at\right]$$

$$= -\frac{1}{2a^3}\left[\sinh at - at\cosh at\right] \text{ (Ans)}$$

No 46 Find $L^{-1}\left\{\frac{s}{(s^2+9)^2}\right\}$ (2 marks)

Soln $L^{-1}\left\{\frac{s}{(s^2+9)^2}\right\} = L^{-1}\left\{\frac{s}{(s^2+3^2)^2}\right\}$

$$= \frac{1}{2\times 3} t\sin 3t = \frac{1}{6} t\sin 3t.$$

HOME TASK

47) Find Inverse Laplace transform of

(a) $\frac{3}{s+3}$ (b) $\frac{2s}{s^2+4}$ (c) $\frac{2s-1}{s^2+8}$

(d) $\frac{s e^{-\pi s}}{s^2+9}$ (e) $\frac{e^{-5s}}{s}$.

No 48 Find $L^{-1}\left\{\frac{3s}{s^2+2s-8}\right\}$

No 49 Find $L^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\}$

No 50) Find $L^{-1}\left\{\frac{s^3}{s^4-a^4}\right\}$

(51) find $L^{-1}\left\{ \frac{1}{s^3-a^3} \right\}$

(52) find $L^{-1}\left\{ \frac{s^2+6}{(s^2+1)(s^2+4)} \right\}$

(53) find $L^{-1}\left\{ \frac{s+2}{(s^2+4s+5)^2} \right\}$

(54) find $L^{-1}\left\{ \frac{s}{s^4+s^2+1} \right\}$

(55) find $L^{-1}\left\{ \frac{1}{s(s+a)^3} \right\}$

(56) find $L^{-1}\left\{ \log\left(\frac{s^2+1}{s(s+1)} \right) \right\}$

(57) find $L^{-1}\left\{ \tan^{-1}\left(\frac{2}{s} \right) \right\}$

(58) find $L^{-1}\left\{ \frac{s}{(s^2+4)^2} \right\}$ (L.T Ref Q No 42)

(59) find $L^{-1}\left\{ \frac{1}{(s^2+9)^2} \right\}$ (L.T Ref Q No 43)

(60) find $L^{-1}\left\{ \frac{s}{(s^2-25)^2} \right\}$ (L.T Ref Q No 44)

(61) find $L^{-1}\left\{ \frac{1}{(s^2-36)^2} \right\}$ (L.T Ref Q No 45)

(62) find $L^{-1}\left\{ \frac{s}{(s^2+16)^2} \right\}$ (2 Mark, Apply F 19)

(63) find $L^{-1}\left\{ \frac{1}{(s^2+9)^2} \right\}$ (2 Mark, Apply F 20)

(64) find $L^{-1}\left\{ \frac{s}{(s^2-4)^2} \right\}$ (2 Mark, Apply F21)

(65) find $L^{-1}\left\{ \frac{1}{(s^2-16)^2} \right\}$ (2 Mark, Apply F22)

(END).

5: FOURIER SERIES

MATH-III/1

Ⓐ PERIODIC FUNCTION

i) A function $f(x)$ is said to be periodic function of period "T" if $\boxed{f(x+T)=f(x), \forall x}$

The smallest +ve period of $f(x)$ is called its primitive period (period)

(ii) The period of $\sin x, \cos x, \sec x$ and $\operatorname{cosec} x$ are all 2π.

But period of $\tan x$ and $\cot x$ are both π

NOTE

(i) If T is a primitive period of $f(x)$, then $nT, n \in Z^+$ is a period of $f(x)$

(ii) If period of $f(x)=T \Rightarrow$ period of $c \cdot f(x)=T$

(iii) If period of $f(x)$ and $g(x)$ are both T then period of $c_1 f(x)+c_2 g(x)$ is T

(iv) If period of $f(x)=\frac{a}{b}$ & $g(x)=\frac{c}{d}$ then period of $f(x)+g(x) = \text{LCM of } \frac{a}{b} \,\&\, \frac{c}{d}$

$$= \frac{\text{LCM of } (a,c)}{\text{GCM of } (b,d)}$$

Ⓑ EVEN FUNCTION

(i) A function $f(x)$ is said to be an Even function if $\boxed{f(-x)=f(x), \forall x}$

(ii) x^2 is an even function for $f(x)=x^2 \Rightarrow f(-x)=(-x)^2=x^2=f(x)$ $\forall x$.

(c) Odd function

i) A function $f(x)$ is said to be an odd function if $\boxed{f(-x) = -x \ \forall x}$

ii) x^3 is an odd function for $f(x) = x^3 \Rightarrow f(-x) = (-x)^3 = -x^3 = -f(x)$ $\forall x$.

NOTES

(i) $\int_{\alpha}^{\alpha+2\pi} \sin(nx)\,dx = \int_{\alpha}^{\alpha+2\pi} \cos nx\,dx = 0$

(ii) $\sin(n\pi) = 0, n \in Z$

$\cos n\pi = (-1)^n, n \in Z$

(iii) $\int_{-a}^{a} f(x)\,dx = 0$, if $f(x)$ odd

$= 2\int_0^a f(x)\,dx$ if $f(x)$ is ~~odd~~ even

(iv) Higher order Integration by parts

$\int(uv)\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \text{------}$

where $v_1 = \int v\,dx, v_2 = \int v_1\,dx$ ------

$u' = \frac{du}{dx}, u'' = \frac{du'}{dx}$ ------

Problems

① Define periodic function [do (i) & (ii)]

(2) Define even function [do (i), (ii)]

(3) Define odd function [do (i) (ii)]

④ Find period of constant function.

Soln Let $f(x) = c$ is a constant function where c is a constant.

Let T be period of $f(x) \Rightarrow f(x+T) = f(x)\ \forall x$

$\Rightarrow c = c$ from (1) which is an identity.

Hence constant function has arbitry period.

5) Find period of $\cos x$.

Soln Let $f(x) = \cos x$ & period of $f(x)$ be T

Then $f(x+T) = f(x), \forall x$

$\Rightarrow \cos(x+T) = \cos x$

$\Rightarrow \cos(x+T) = \cos(x+2\pi)$

$\Rightarrow x+T = x+2\pi \Rightarrow T = 2\pi$

$\therefore$ period of $\cos x$ is 2π.

6) Find period of $\tan 3x$.

Soln Let $f(x) = \tan 3x$ & period of $f(x)$ be T

Then $f(x+T) = f(x), \forall x \Rightarrow \tan 3(x+T) = \tan 3x$

$\Rightarrow \tan 3(x+T) = \tan(3x+\pi)$

$\Rightarrow 3(x+T) = 3x+\pi \Rightarrow 3x+3T = 3x+\pi$

$\Rightarrow 3T = \pi \Rightarrow T = \pi/3$

$\therefore$ period of $\tan 3x$ is $\pi/3$.

7) Find period of $\sin 3x + 5\sin\left(\frac{x}{2}\right)$

Soln Let $f(x) = \sin 3x$ & its period be T_1

Then $f(x+T_1) = f(x), \forall x$.

$\Rightarrow \sin 3(x+T_1) = \sin 3x \Rightarrow \sin(3x+3T_1) = \sin(3x+2\pi)$

$\Rightarrow 3x+3T_1 = 3x+2\pi \Rightarrow 3T_1 = 2\pi \Rightarrow T_1 = \frac{2\pi}{3}$

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Let $g(x) = 5\sin(\frac{x}{2})$ and T_2 be its period.

Then $g(T_2+x) = g(x), \forall x \Rightarrow 5\sin(\frac{x+T_2}{2}) = 5\sin(\frac{x}{2})$

$\Rightarrow \sin(\frac{x+T_2}{2}) = \sin(\frac{x}{2}+2\pi) \Rightarrow \frac{x+T_2}{2} = \frac{x}{2}+2\pi$

$\Rightarrow \frac{x+T_2}{2} = \frac{x+4\pi}{2} \Rightarrow x+T_2 = x+4\pi \Rightarrow T_2 = 4\pi$

Now period of $\sin 3x + 5\sin(\frac{x}{2})$

$=$ period of $\{f(x)+g(x)\}$

$=$ LCM of T_1 & T_2 $=$ LCM of $(\frac{2\pi}{3}, 4\pi)$

$= \frac{\text{LCM of } (2\pi, 4\pi)}{\text{GCM of } (3, 1)} = \frac{4\pi}{1} = 4\pi$ (Ans)

8) Show that constant function is an even function

Soln: Let $f(x) = c$ is a constant function where c is a constant.

Now $f(-x) = c = f(x)\ \forall x$

~~Hence~~ $\Rightarrow f(x)$ is an even function.

(9) Show that $\tan x$ is an odd function.

Soln Let $f(x) = \tan x$ —— (1)

Now $f(-x) = \tan(-x) = -\tan x = -f(x), \forall x$

Hence $f(x)$ is an odd function

i.e $\tan x$ is an odd function.

HOME TASK

(10) Find period of following:

(a) $\sin x$ (b) $\sin 5x$ (c) $\cos(\frac{x}{3})$ (d) $\tan 4x$

(e) $\sec x$ (f) $\operatorname{cosec} 2x$ (g) $\sin 4x + 2\tan 3x$

(11) Show that x^4+x^2+1 is an even function

(12) Show that $\sin x - 1$ is an odd function

(13) show that $\sec x$ is an ~~odd~~ even function.

(D) FOURIER SERIES:

(i) The fourier series of a function in $\alpha \leq x \leq \alpha + 2\pi$ is a Trigonometrical series of the form

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx}$$

where a_0, a_n, b_n are called fourier co-efficients.

(ii)
$$a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx$$
$$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x).\cos nx\,dx$$
$$b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx$$

The formula for values of a_0, a_n & b_n is called **Euler's formula**

(E) DIRICHLET'S CONDITION

Any function $f(x)$ can be expressed as Fourier series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ if

(i) $f(x)$ is periodic, single valued and finite.

(ii) $f(x)$ has a finite number of finite dis-continuity in any one period.

(iii) $f(x)$ has at most finite number of maxima and minima.

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QNO13 Define Fourier series of $f(x)$ in $\alpha \le x \le \alpha + 2\pi$

Hints: write down (i) and (ii) of ~~FOURIER SERIES~~ (D)

QNO14 State Euler's formula for Fourier series of $f(x)$ in $\alpha \le x \le \alpha + 2\pi$.

Hints: write down (i) and (ii) of (D)

(NO15) Define Fourier series of $f(x)$ in $0 \le x \le 2\pi$.

Hints: write down (i) & (ii) of D, $\alpha = 0$

NO16) State Euler's formula for fourier series of $f(x)$ in $0 \le x \le 2\pi$

Hints: write down (i), (ii) of D, $\alpha = 0$

NO17) State Dirichlet's condition for fourier series of a function $f(x)$.

Hints: write down (E)

NO18) obtain fourier series of $f(x) = e^{-x}$ in $0 < x < 2\pi$

Sol^n Let fourier series of $f(x) = e^{-x}$ in $0 < x < 2\pi$ be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{---(1)}$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{\pi}\left[\frac{e^{-x}}{-1}\right]_0^{2\pi}$$

$$= -\frac{1}{\pi}\left[e^{-x}\right]_0^{2\pi} = -\frac{1}{\pi}\left[e^{-2\pi} - 1\right] = \frac{1 - e^{-2\pi}}{\pi} \quad \text{---(2)}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{(-1)^2 + n^2}\left((-1)\cos nx + n\sin nx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{n^2+1}\left(n\sin nx - \cos nx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-2\pi}}{n^2+1}(-1) - \frac{1}{n^2+1}(-1)\right]$$

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$$= \frac{1}{\pi}\left\{\frac{1}{n^2+1} - \frac{e^{-2\pi}}{n^2+1}\right\} = \frac{1-e^{-2\pi}}{\pi(n^2+1)} \quad \text{——} \ (3)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{(-1)^2+n^2}\left((-1)\sin nx - n\cos nx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{n^2+1}\left(-\sin nx - n\cos nx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-2\pi}}{n^2+1}(-n) - \frac{1}{n^2+1}(-n)\right]$$

$$= \frac{1}{\pi}\left[\frac{n}{n^2+1} - \frac{n e^{-2\pi}}{n^2+1}\right]$$

$$= \frac{n(1-e^{-2\pi})}{\pi(n^2+1)} \quad \text{——} \ (4)$$

Now $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$\Rightarrow e^{-x} = \frac{1-e^{-2\pi}}{2\pi} + \sum_{n=1}^{\infty}\frac{1-e^{-2\pi}}{\pi(n^2+1)}\cos nx + \sum_{n=1}^{\infty}\frac{n(1-e^{-2\pi})}{\pi(n^2+1)}\sin nx$$

$$\Rightarrow e^{-x} = \frac{1-e^{-2\pi}}{2\pi} + \frac{1-e^{-2\pi}}{\pi}\sum\frac{1}{n^2+1}\cos nx + \frac{1-e^{-2\pi}}{\pi}\sum\frac{n}{n^2+1}\sin nx$$

$$e^{-x} = \frac{1-e^{-2\pi}}{\pi}\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{n^2+1}\cos nx + \sum_{n=1}^{\infty}\frac{n}{n^2+1}\sin nx\right\}$$

$$\Rightarrow e^{-x} = \frac{1-e^{-2\pi}}{\pi}\left\{\frac{1}{2} + \left(\frac{1}{2}\cos x + \frac{1}{5}\cos 2x + \cdots\right) + \left(\frac{1}{2}\sin x + \frac{2}{5}\sin 2x + \cdots\right)\right\}$$

which is the required Fourier series for e^{-x} in $0 < x < 2\pi$ (Ans.)

(19) obtain Fourier series for $f(x) = x - x^2$ in $-\pi < x < \pi$ and hence deduce that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$

Soln Let Fourier series of $f(x) = x - x^2$ in $-\pi < x < \pi$ be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{——— (1)}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\,dx = \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi^2}{2} - \frac{\pi^3}{3}\right) - \left(\frac{\pi^2}{2} + \frac{\pi^3}{3}\right)\right] = \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^3}{3} - \frac{\pi^2}{2} - \frac{\pi^3}{3}\right]$$

$$= \frac{1}{\pi}\left(-\frac{2\pi^3}{3}\right) = -\frac{2\pi^2}{3} \quad \text{——— (2)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\left(\frac{\sin nx}{n}\right) - (1-2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(x-x^2)(\sin nx)}{n} + \frac{(1-2x)\cos nx}{n^2} + \frac{2\sin nx}{n^3}\right]_{-\pi}^{\pi}$$

$$\frac{1}{\pi}\left[\frac{(1-2\pi)(-1)^n}{n^2} - \frac{(1+2\pi)(-1)^n}{n^2}\right] = \frac{(-1)^n}{n^2\pi}\left[(1-2\pi)-(1+2\pi)\right]$$

$$= \frac{(-1)^n}{n^2\pi}\left[1-2\pi-1-2\pi\right] = \frac{(-1)^n}{n^2\pi}\times(-4\pi) = \frac{-4(-1)^n}{n^2} \quad \text{—— (3)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\left(-\frac{\cos nx}{n}\right) - (1-2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-(x-x^2)\cos nx}{n} + \frac{(1-2x)\sin nx}{n^2} - \frac{2\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{\frac{-(\pi-\pi^2)(-1)^n}{n} - \frac{2(-1)^n}{n^3}\right\} - \left\{\frac{-(-\pi-\pi^2)(-1)^n}{n} - \frac{2(-1)^n}{n^3}\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{(\pi^2-\pi)(-1)^n}{n} - \frac{2(-1)^n}{n^3} - \frac{(\pi+\pi^2)(-1)^n}{n} + \frac{2(-1)^n}{n^3}\right]$$

$$= \frac{(-1)^n}{\pi\, n}\left[\pi^2-\pi-\pi-\pi^2\right] = \frac{(-1)^n}{n\pi}(-2\pi) = \frac{-2(-1)^n}{n} \quad \text{—— (4)}$$

Now from (1)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\Rightarrow x - x^2 = -\frac{2\pi^2}{6} + \sum_{n=1}^{\infty}\frac{-4(-1)^n}{n^2}\cos nx + \sum \frac{-2(-1)^n}{n}\sin nx$$

which is the required fourier series of $f(x)$. —— (5)

Now put $x=0$ in (5)

$$\Rightarrow 0 = -\frac{2\pi^2}{6} + \sum_{n=1}^{\infty} \frac{(-4)(-1)^n}{n^2} \times 1$$

$$\Rightarrow 0 = -\frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\Rightarrow 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

$$\Rightarrow -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\cdots = -\frac{\pi^2}{12}$$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots = \frac{\pi^2}{12} \quad \square$$

HOME TASK:

(20) Find a_0 if $f(x) = x - x^2$, $-\pi < x < \pi$.

(21) Find $2a_0$ if $f(x) = x$ in $0 \le x \le 2\pi$

(22) If fourier series of $f(x) = e^{-x}$ in $0 < x < 2\pi$ is

$$e^{-x} = \frac{a_0}{2} + \frac{1-e^{-2\pi}}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2+1} \cos nx + \frac{1-e^{-2\pi}}{\pi} \sum \frac{n}{n^2+1} \sin nx$$

then find a_0.

(F) fourier series for function having point of discontinuity:

(i) Let $f(x)$ is a function defined in the interval $(\alpha, \alpha+2\pi)$ by

$$f(x) = \begin{cases} \phi_1(x), & \alpha < x < c \\ \phi_2(x), & c < x < \alpha+2\pi \end{cases} \quad \text{--- (1)}$$

point c is called point of discontinuity

$$a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx = \frac{1}{\pi}\left\{\int_{\alpha}^{c} \phi_1(x)\,dx + \int_{c}^{\alpha+2\pi} \phi_2(x)\,dx\right\}$$

$$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left\{\int_{\alpha}^{c} \phi_1(x)\cos nx\,dx + \int_{c}^{\alpha+2\pi} \phi_2(x)\cos nx\,dx\right\}$$

$$b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left\{\int_{\alpha}^{c} \phi_1(x)\sin nx\,dx + \int_{c}^{\alpha+2\pi} \phi_2(x)\sin nx\,dx\right\}$$

(ii) At point of finite discontinuity $x = c$, there is a finite jump in the graph of function

at $x = c$

$$\boxed{f(x) = \frac{1}{2}\left[f(c-) + f(c+)\right]}$$

No 33 Find Fourier series of $f(x)$ if

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

Hence deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{8}$$

Soln

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases} \quad \text{—— (1)}$$

Let Fourier series of $f(x)$ be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—— (2)}$$

Now $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right\}$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0}(-\pi)\,dx + \int_{0}^{\pi} x\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[-\pi x\right]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{-\pi^2 + \frac{\pi^2}{2}\right\} = \frac{1}{\pi}\left\{-\frac{\pi^2}{2}\right\} = -\frac{\pi}{2} \quad \text{—— (3)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} -\pi\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[-\frac{\pi\sin nx}{n}\right]_{-\pi}^{0} + \left[x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right\} = \frac{(-1)^n - 1}{n^2\pi} \quad \text{—— (4)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} -\pi\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{\pi\cos nx}{n}\right]_{-\pi}^{0} + \left[(x)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{\pi}{n} - \frac{\pi(-1)^n}{n}\right] + \left[\frac{-\pi(-1)^n}{n}\right]\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\pi}{n} - \frac{\pi(-1)^n}{n} - \frac{\pi(-1)^n}{n}\right\} = \frac{1}{\pi}\left\{\frac{\pi}{n} - \frac{2\pi(-1)^n}{n}\right\}$$

$$= \frac{1 - 2(-1)^n}{n} \quad \text{—— (5)}$$

Now from (2)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$\Rightarrow f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2\pi}\cos nx + \sum_{n=1}^{\infty}\frac{1-2(-1)^n}{n}\sin nx \quad \text{—— (6)}$$

$$\Rightarrow \cancel{f(x) = -\frac{\pi}{4} + \frac{1}{\pi}\left\{-\frac{2}{1^2}\cos x - \frac{2}{3^2}\cos 3x\right.}$$

which is the required fourier series.

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Now $x=0$ is the point of discontinuity.

Hence $f(0) = \frac{1}{2}\{f(0+) + f(0-)\}$

$$= \frac{1}{2}\{0 + (-\pi)\} = -\frac{\pi}{2}.$$

put $x=0$ in eqn(6) $\Rightarrow$

$$-\frac{\pi}{2} = -\frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2 \pi} \times 1$$

$$\Rightarrow -\frac{\pi}{2} + \frac{\pi}{4} = \frac{1}{\pi}\left\{-\frac{2}{1^2} - \frac{2}{3^2} - \frac{2}{5^2} - \cdots\right\}$$

$$\Rightarrow -\frac{\pi}{4} = -\frac{2}{\pi}\left\{\frac{1}{1^2} + \frac{1}{3^2} + \cdots\right\}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = -\frac{\pi}{4} \times \frac{\pi}{-2} = \frac{\pi^2}{8} \text{ (Proved)}$$

(24) If $f(x) = 1 - \frac{2x}{\pi}, \; -\pi \leq x \leq 0$

$1 + \frac{2x}{\pi}, \; 0 \leq x \leq \pi$ then find $f(0)$.

Soln

$$f(x) = \begin{cases} 1 - \frac{2x}{\pi}, & -\pi \leq x < 0 \\ 1 + \frac{2x}{\pi}, & 0 \leq x \leq \pi \end{cases} \quad \text{--- (1)}$$

Here $x=0$ is a point of discontinuity.

Hence $f(0) = \frac{1}{2}\{f(0+) + f(0-)\}$

$$= \frac{1}{2}\{1 + 1\} = \frac{1}{2} \times 2 = 1 \text{ (Ans)}$$

(25) Find Fourier series of $f(x)$ defined by

$$f(x) = x, \; 0 \leq x \leq \pi$$
$$2\pi - x, \; \pi \leq x \leq 2\pi$$

and hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

28 10 11 12 13 14 15 16 33 14 15 16 17 18 19 20
29 17 18 19 20 21 22 23 34 21 22 23 24 25 26 27
30 24 25 26 27 28 29 30 35 28 29 30 31

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(26) If $f(x) = -\pi, \; -\pi < x < 0$
$x, \; 0 < x < \pi$ then find $f(\pi/2)$

(27) If $f(x) = \pi, \; -\pi < x < 0$
$-x, \; 0 < x < \pi$ then find $f(0)$.

(H) FOURIER SERIES OF EVEN & ODD FUNCTION:

NOTE: (i) An even function is symmetrical about Y-axis.
(ii) An odd function is symmetrical about ~~x-axis~~. origin

$(-\pi \quad \pi)$

(i) For even function

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

Hence it doesnot contain sine terms

(ii) For odd function

$$a_0 = 0, \quad a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

So it does not contain cosine term.

(38) Express $f(x) = x$ as fourier series in $-\pi < x < \pi$.

Soln $f(x) = x$ in $(-\pi, \pi)$ —— (1)

Here $f(-x) = -x = -f(x)\ \forall x$. Hence $f(x)$ is an odd function.

Thus $a_0 = a_n = 0$. —— (2)

Now $b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^\pi x\sin(nx)\,dx$

$$= \frac{2}{\pi}\left[(x)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[-\frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_0^\pi = \frac{2}{\pi}\left[\frac{-\pi(-1)^n}{n} - 0\right]$$

$$= \frac{2}{\pi}\times\frac{-\pi(-1)^n}{n} = \frac{-2(-1)^n}{n} \quad \text{—— (3)}$$

Hence fourier series of $f(x)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$\Rightarrow x = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n}\sin(nx) = -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(nx)$$

$$\Rightarrow x = -2\left\{-\frac{1}{1}\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x - \cdots\right\}$$

$$\Rightarrow x = 2\left\{\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots\right\}$$

which is the required fourier series for $f(x)$ (Ans)

(29) For a function $f(x)$ defined by $f(x) = |x|$, $-\pi < x < \pi$, obtain Fourier series. Deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

30) prove that $x\cos x = \frac{1}{2}\sin x + 2\sum_{n=2}^{\infty} \frac{n(-1)^n}{n^2-1}\sin nx$ in $-\pi < x < \pi$.

(31) show that

$$\sin ax = \frac{2\sin a\pi}{\pi}\left[\frac{\sin x}{1^2-a^2} - \frac{2\sin 2x}{2^2-a^2} + \cdots\right].$$

((END.))