INDEX

Name MR. SUDHAKAR SINGH Subject ENGG.MATH-II CTH-20

STD. ______ Div. ______ Roll No. ______ School PNS.

Sr. No.	Page No.	Title	Date	Teacher's Sign / Remarks
01		UNIT-I : DETERMINANTS AND MATRICES:		
02		UNIT-II: INTEGRAL CALCULUS		
03		UNIT-III: CO-ORDINATE GEOMETRY		
04		UNIT-IV: VECTOR ALGEBRA		
05		UNIT-V : DIFFERENTIAL EQUATIONS:		

A: DETERMINANT: It is defined as a scalar value associated with a square matrix.

- The order of the square matrix is called the order of the determinant associated with the square matrix.
- If 'A' is a square matrix of order 'n' (n Rows and n columns) then determinant of 'A' is denoted by det.A or $|A|$ or Δ. Here $|A|$ is an nth order determinant.

A1: 1st. ORDER DETERMINANT: The Determinant Associated with 1st order square matrix is called 1st order determinant. It is Trival determinant. Defined as follows:

If $A = [a]_{1\times1} \Rightarrow |A| = |a| = a$.

$\therefore \boxed{|a| = a}$ — FORMULA.

Ex: $|0| = 0$, $|1| = 1$, $|-3| = -3$, $|\frac{1}{2}| = \frac{1}{2}$ so on.

A2: 2ND. ORDER DETERMINANT:

Let us consider the system of equations

$$a_1x + b_1 = 0 \quad \text{——(1)}$$
$$a_2x + b_2 = 0 \quad \text{——(2)}$$

From eqn (1) $a_1x = -b_1 \Rightarrow x = -\frac{b_1}{a_1}$

If Eqn (1) & (2) posses same solution then $x = -\frac{b_1}{a_1}$ must satisfies eqn (2). Hence

$$a_2x + b_2 = 0 \Rightarrow -\frac{a_2b_1}{a_1} + b_2 = 0$$
$$\Rightarrow \quad -a_2b_1 + a_1b_2 = 0 \quad \text{i.e} \quad a_1b_2 - a_2b_1 = 0$$

The expression $a_1b_2 - a_2b_1$ is called the value of the 2nd. order determinant associated with the second order square matrix $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$

$$\therefore \boxed{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1}$$ — FORMULA.

Sign system: $\begin{array}{c|cc} & C_1 & C_2 \\ \hline R_1 & + & - \\ R_2 & - & + \end{array}$

Note that a_1, a_2, b_1, b_2 are called elements or constituents of the determinants.

Ex: $\begin{vmatrix} 1 & -3 \\ 2 & 4 \end{vmatrix} = (1)(4) - (2)(-3) = 4 + 6 = 10$

A3: DETERMINANT OF 3RD. ORDER:

Let us consider the system of Equations

$$a_1x + b_1y + c_1 = 0 \quad \text{——(1)}$$
$$a_2x + b_2y + c_2 = 0 \quad \text{——(2)}$$
$$a_3x + b_3y + c_3 = 0 \quad \text{——(3)}$$

Solving Equation ① & ② by cross-multiplication we have

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \quad y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

If Equation ① ② & ③ posses same solution then above values of x and y will satisfy eqⁿ (3). Thus we have

$$a_3 \cdot \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} + b_3 \cdot \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} + c_3 = 0$$

$$\Rightarrow a_3(b_1c_2 - b_2c_1) + b_3(c_1a_2 - c_2a_1) + c_3(a_1b_2 - a_2b_1) = 0$$

$$\Rightarrow a_3b_1c_2 - a_3b_2c_1 + b_3c_1a_2 - b_3c_2a_1 + c_3a_1b_2 - c_3a_2b_1 = 0$$

$$\Rightarrow a_1(b_2c_3 - b_3c_2) - b_1(c_3a_2 - c_2a_3) + c_1(a_2b_3 - a_3b_2) = 0$$

The expression on LHS is called the determinant of 3rd order of the system. denoted by

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\therefore \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(c_3a_2 - c_2a_3) + c_1(a_2b_3 - a_3b_2)$$

$$= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

FORMULA.

- **SIGN SYSTEM:**

$$\begin{array}{cl} \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} & \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3. \end{array} \\ \begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ C_1 & C_2 & C_3 \end{array} & \end{array}$$

NOTE: 1

- No of Elements in an nth. order determinant $= n^2$.
- No of terms in an nth o determinant after expanding $= \lfloor n$.
- Element belonging to ith row and jth column of a determinant is denoted by a_{ij}. Thus the general 3rd order determinant is

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

SARRUS METHOD:

It is a mnemonic device to get the value of a 3rd order determinant as follows:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix}$$

(-ve) (-ve) (-ve)

+ve +ve +ve

Writing 1st. two columns on RHS of the determinant

$$= (a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3) - (a_3b_2c_1 + b_3c_2a_1 + c_3a_2b_1)$$

B: MINOR AND COFACTOR:

- Minor of any element a_{ij} of a determinant is denoted by M_{ij} and defined by it is the determinant obtained by suppressing elements in ith row and jth column.

  Ex: Minor of $a_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

- Co-factor of any element a_{ij} of a determinant is denoted by C_{ij} and defined by $\boxed{C_{ij} = (-1)^{i+j} M_{ij}}$ - Formula.

  Ex: Cofactor of $a_{12} = C_{12} = (-1)^{1+2} M_{12} = -M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

  We see that $C_{11} = M_{11}, C_{12} = -M_{12}, C_{13} = M_{13}$

  $C_{21} = -M_{21}, C_{22} = M_{22}, C_{23} = -M_{23}$

  $C_{31} = M_{31}, C_{32} = -M_{32}, C_{33} = M_{33}$.

NOTE: 2 • The sum of the product of elements of any row (column) with their corresponding co-factors is equal to the value of the determinant i.e

$$\Delta = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} \text{ so on.}$$

- The sum of product of elements of any row (column) with corresponding cofactors of any other row (column) is zero i.e

$$a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = a_{11}C_{13} + a_{21}C_{23} + a_{31}C_{33} = \cdots = 0$$

C: PROPERTIES OF DETERMINANT:

- The values of a determinant remains unchanged if rows are written as columns and vice-versa.
- If two adjacent rows (or columns) of a determinant are interchanged, then the sign of the determinant changes keeping its absolute value fixed.
- If each elements in any row (or column) is multiplied by same factor then the determinant is multiplied by that factor.
- If two rows (or columns) of a determinant are identical (proportional) then the determinant vanishes.
- If each elements of any row (or column) consist of two or more than two terms, then the determinant can be expressed as sum of two or more determinants.
- If all elements of any row (or column) be increased or decreased by any equimultiple of the corresponding element of one or more of the other rows (column) then the value of the determinant remains unchanged.

$R_i \to R_i + kR_j$, $C_i \to C_i + kC_j$, $R_i \leftrightarrow R_j$, $C_i \leftrightarrow C_j$

$i \neq j$.

Note that all rows (or columns) should not be replaced simultaneously.

- If all elements of any row (or column) of a determinant are zero then the value of the determinant is zero
- If each element of a determinant Δ is a polynomial in x such that $\Delta = 0$ for $x = a$ then $x-a$ is a factor of Δ. This is known as factor theorem. In otherword if two rows (or columns) of Δ become identical for $x=a$ then $x-a$ is a factor of Δ. Note that if r rows (or columns) become identical for $x=a$ then $(x-a)^{r-1}$ is a factor of Δ.

D: DETERMINANT MULTIPLICATION:

Lets consider two 2nd. order determinant $\Delta_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}$ then

Method-i (Row by column multiplication)

$$\Delta_1 \cdot \Delta_2 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1l_1 + b_1l_2 & a_1m_1 + b_1m_2 \\ a_2l_1 + b_2l_2 & a_2m_1 + b_2m_2 \end{vmatrix}$$

Method-II: (Row by Row multiplication)

$$\Delta_1 \cdot \Delta_2 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1l_1 + b_1m_1 & a_1l_2 + b_1m_2 \\ a_2l_1 + b_2m_1 & a_2l_2 + b_2m_2 \end{vmatrix}$$

Method-III: (Column by Column multiplication)

$$\Delta_1 \cdot \Delta_2 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1l_1 + a_2l_2 & a_1m_1 + a_2m_2 \\ b_1l_1 + b_2l_2 & b_1m_1 + b_2m_2 \end{vmatrix}$$

Method-IV: (Column by Row multiplication)

$$\Delta_1 \cdot \Delta_2 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1l_1 + a_2m_1 & a_1l_2 + a_2m_2 \\ b_1l_1 + b_2m_1 & b_1l_2 + b_2m_2 \end{vmatrix}$$

Similarly two 3rd order determinants can be multiplied.

E: TYPE OF SYSTEM OF LINEAR EQUATIONS:

SYSTEM OF LINEAR EQUATIONS

- INCONSISTENT (System having no solution)
- CONSISTENT (system having solution)
  - (System having (OR) only one solution)
  - (System having Infinite No. of Solutions)
- Homogeneous (If all constant terms are zero)
- Non-homogeneous (If atleast one constant is not zero)

GENERAL FORM:

- $\left.\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}\right\}$ In two variables.
- $\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\}$ In three variables.

TRIVAL & NONTRIVAL SOLUTIONS:

If the values of all variables are zero for which the given system of linear equations are satisfied then this solution is called Trival solution. Otherwise it is called non-trival solution

NOTE: If A, B are square matrices of order n then

(i) $|AB| = |A| \cdot |B|$

(ii) $|KA| = K^n |A|$, K scalar.

F: CRAMER'S RULE (DETERMINANT METHOD)

FOR SYSTEM OF TWO VARIABLES	FOR SYSTEM OF THREE VARIABLES
• Given system of equations are $a_1x+b_1y=c_1$ $a_2x+b_2y=c_2$ $\Big\\}$ —— ① • Determinant of the system $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$ • Find Δ_1, Δ_2 by $\Delta_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ $\Delta_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ • Find x and y by $x = \frac{\Delta_1}{\Delta}$, $y = \frac{\Delta_2}{\Delta}$	• Given system of equations are $a_1x+b_1y+c_1z=d_1$ $a_2x+b_2y+c_2z=d_2$ $a_3x+b_3y+c_3z=d_3$ $\Big\\}$ —— ① • Determinant of the system $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$ • Find $\Delta_1, \Delta_2, \Delta_3$ by $\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$ $\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ $\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$ • Find x, y, z by $x = \frac{\Delta_1}{\Delta}$, $y = \frac{\Delta_2}{\Delta}$, $z = \frac{\Delta_3}{\Delta}$

G: NATURE OF SOLUTION

FOR SYSTEM OF TWO VARIABLES:	FOR SYSTEM OF THREE VARIABLES:
• If $\Delta \neq 0$ but atleast one of Δ_1 Δ_2 is not zero ($\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$) $\Rightarrow$ system is consistent and has unique non trival solution. (Intersecting lines) • $\Delta \neq 0$ & $\Delta_1 = \Delta_2 = 0$ ($\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$) $\Rightarrow$ system is consistent and has trival solution. (Intersecting lines). • If $\Delta = \Delta_1 = \Delta_2 = 0$ ($\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$) $\Rightarrow$ system is consistent and has infinite solutions (Identical lines or concident lines) • If $\Delta = 0$ but at least one of Δ_1, Δ_2 is not zero ($\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$) $\Rightarrow$ system is inconsistent (parallel lines)	• If $\Delta \neq 0$ and Atleast one of $\Delta_1, \Delta_2, \Delta_3$ is not zero $\Rightarrow$ system is consistent and has unique non trival solution (Intersecting planes) • $\Delta \neq 0$ & $\Delta_1 = \Delta_2 = \Delta_3 = 0 \Rightarrow$ system is consistent and has trival solution (Intersecting planes) • If $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0 \Rightarrow$ The system is consistent and has infinite solutions. (Identical planes) • If $\Delta = 0$ but at least one of $\Delta_1, \Delta_2, \Delta_3$ is not zero $\Rightarrow$ system is inconsistent (Parallel planes) NOTE: For the system $ax+by+cz=d_1$, $ax+by+cz=d_2$ and $ax+by+cz=d_3$ (Atleast two of d_1, d_2, d_3 are not equal) then $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$ But the system is inconsistent because the system represents three parallel planes.

NOTE:

- For homogeneous system of linear equations $\Delta_1=\Delta_2=\Delta_3=0$. Hence if $\Delta\neq 0 \Rightarrow$ The system posseses unique trival solutions ($x=0, y=0, z=0$) (7)

If $\Delta=0 \Rightarrow$ The system posseses infinite solutions

- If the system of equations $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$, $a_3x+b_3y+c_3=0$ is consistent

then $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$ but the converse is not true.

- For the system $a_1x+b_1y+c_1z=0$, $a_2x+b_2y+c_2z=0$, $a_3x+b_3y+c_3z=0$ Eliminating x, y, z we have

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Ex:1: calculate x if $\begin{vmatrix} x & 3 \\ 4 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix}$

Soln: $\begin{vmatrix} x & 3 \\ 4 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} \Rightarrow (x)(2)-(4)(3) = (3)(2)-(1)(2)$

$\Rightarrow 2x-12 = 6-2 \Rightarrow 2x-12=4 \Rightarrow 2x = 4+12=16$

$\Rightarrow x = \frac{16}{2} = 8$ (Ans)

Ex:2 Evaluate $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 3 & 3 \\ 2 & -1 & 1 \end{vmatrix}$

Soln: $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 3 & 3 \\ 2 & -1 & 1 \end{vmatrix} = 1\begin{vmatrix} 3 & 3 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix}$

$= 1(3+3)-2(-1-6)+3(1-6) = 6+14-15 = 5$ (Ans)

(OR)

$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & 3 & 3 \\ 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 3 & 3 \\ 2 & -1 & 1 \end{vmatrix}\begin{matrix} 1 & 2 \\ -1 & 3 \\ 2 & -1 \end{matrix}$$ by Sarous method

$= (3+12+3)-(18-3-2) = 18-13 = 5$ (Ans)

Ex-3: Find minors and co-factors of elements $-3, 5, -1$ in the determinant $\begin{vmatrix} 1 & -3 & 2 \\ 3 & 0 & 5 \\ -1 & 1 & 7 \end{vmatrix}$

Soln: Minor of $-3 = M_{12} = \begin{vmatrix} 3 & 5 \\ -1 & 7 \end{vmatrix} = 21+5=26$

cofactor of $-3 = C_{12} = -M_{12} = -26$

Minor of $5 = M_{23} = \begin{vmatrix} 1 & -3 \\ -1 & 1 \end{vmatrix} = 1-3 = -2$

co-factor of $5 = C_{23} = -M_{23} = 2$

Minor of $(-1) = M_{31} = \begin{vmatrix} -3 & 2 \\ 0 & 5 \end{vmatrix} = -15-0 = -15$

co-factor of $(-1) = C_{31} = M_{31} = -15$

Ex-4: Show that $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$

$$LHS = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \times \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0\times0+c\times c+b\times b & 0\times c+c\times0+b\times a & 0\times b+c\times a+b\times0 \\ c\times0+0\times c+a\times b & c\times c+0\times0+a\times a & c\times b+0\times a+a\times0 \\ b\times0+a\times c+0\times b & b\times c+a\times0+0\times a & b\times b+a\times a+0\times0 \end{vmatrix}$$

(Row by column multiplication)

$$= \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix} = RHS \ \square$$

Ex-5: Solve the following system of equations by Cramer's rule. Comment on their consistency. $x+y=5$, $3x-2y=7$.

Soln: Given system is $\left.\begin{matrix} x+y=5 \\ 3x-2y=7 \end{matrix}\right\}$ —— ①

Here $\Delta = \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = (1)(-2)-(3)(1) = -2-3 = -5 \neq 0$

$\Delta_1 = \begin{vmatrix} 5 & 1 \\ 7 & -2 \end{vmatrix} = (5)(-2)-(7)(1) = -10-7 = -17$.

$\Delta_2 = \begin{vmatrix} 1 & 5 \\ 3 & 7 \end{vmatrix} = (1)(7)-(3)(5) = 7-15 = -8$

$x = \frac{\Delta_1}{\Delta} = \frac{-17}{-5} = \frac{17}{5}$, $y = \frac{\Delta_2}{\Delta} = \frac{-8}{-5} = \frac{8}{5}$

$\therefore \ x = \frac{17}{5}$, $y = \frac{8}{5}$

Here $\Delta \neq 0$ & $\Delta_1 \neq 0$, $\Delta_2 \neq 0$.

Hence the given system is consistent and has unique non trivial solution

Ex-6: Solve following system of linear equations by determinant method and comment on their consistency. $x+2y+z=7$, $2x+4y+5z=8$, $3x+y+9z=6$

Soln: Given system is $\left.\begin{matrix} x+2y+z=7, \\ 2x+4y+5z=8 \\ 3x+y+9z=6 \end{matrix}\right\}$ —— ①

Here

$$\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 3 & 1 & 9 \end{vmatrix} = 1\begin{vmatrix} 4 & 5 \\ 1 & 9 \end{vmatrix} - 2\begin{vmatrix} 2 & 5 \\ 3 & 9 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix}$$

$$= 1(36-5) - 2(18-15) + 1(2-12)$$

$$= 31-6-10 = 15 \neq 0$$

$$\Delta_1 = \begin{vmatrix} 7 & 2 & 1 \\ 8 & 4 & 5 \\ 6 & 1 & 9 \end{vmatrix} = 7(36-5)-2(72-30)+1(8-24) = 217-84-16 = 117.$$

$$\Delta_2 = \begin{vmatrix} 1 & 7 & 1 \\ 2 & 8 & 5 \\ 3 & 6 & 9 \end{vmatrix} = 1(72-30)-7(18-15)+1(12-24) = 42-21-12 = 9$$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 7 \\ 2 & 4 & 8 \\ 3 & 1 & 6 \end{vmatrix} = 1(24-8)-2(12-24)+7(2-12) = 16+24-70 = -30$$

$$x = \frac{\Delta_1}{\Delta} = \frac{117}{15} = \frac{39}{5}$$

$$y = \frac{\Delta_2}{\Delta} = \frac{9}{15} = \frac{3}{5}$$

$$z = \frac{\Delta_3}{\Delta} = -\frac{30}{15} = -2$$

$\therefore\ x = \frac{39}{15},\ y = \frac{3}{5},\ z = -2.$

Here $\Delta \neq 0$ and atleast one of $\Delta_1, \Delta_2, \Delta_3$ is not zero. so the system is consistent and has unique non trival solutions.

Ex-7: Find the nature of solution for the given system of equations
$x+y+3z=3,\ 2x+2y+4z=4,\ 3x+3y+5z=0$

Sol^n: Given system is
$$\left.\begin{aligned} x+y+3z&=3 \\ 2x+2y+4z&=4 \\ 3x+3y+5z&=0 \end{aligned}\right\} \quad \text{—— (1)}$$

Here $\Delta = \begin{vmatrix} 1 & 1 & 3 \\ 2 & 2 & 4 \\ 3 & 3 & 5 \end{vmatrix} = 0$ Since C_1 and C_2 are identical.

$\Delta_1 = \begin{vmatrix} 3 & 1 & 3 \\ 4 & 2 & 4 \\ 0 & 3 & 5 \end{vmatrix} = 3\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} - 4\begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix}$ Expanding w.r.t C_1

$= 3(10-12)-4(5-9) = -6+16 = 10$

$\Delta = 0$ and $\Delta_1 \neq 0$. Hence Given system is inconsistent i.e it has no solution.

Ex-8: prove that $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = xyz\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$

$$\text{LHS} = \begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = \begin{vmatrix} x & 0 & 1 \\ -y & y & 1 \\ 0 & -z & 1+z \end{vmatrix} \qquad \begin{aligned} C_1 &\to C_1 - C_2 \\ C_2 &\to C_2 - C_3 \end{aligned}$$

$= x\begin{vmatrix} y & 1 \\ -z & 1+z \end{vmatrix} + 1\begin{vmatrix} -y & y \\ 0 & -z \end{vmatrix}$ Expanding w.r.t R_1

$= x\{y(1+z)-(-z)(1)\} + 1\{(-y)(-z)-(0)(y)\}$

$= x(y+yz+z) + yz = xy+xyz+xz+yz$

$= xyz+xy+yz+zx = xyz\left(1+\frac{xy}{xyz}+\frac{yz}{xyz}+\frac{zx}{xyz}\right)$

$= xyz\left(1+\frac{1}{z}+\frac{1}{x}+\frac{1}{y}\right) = xyz\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = \text{RHS}\ \square.$

Ex-9: prove that $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$.

LHS $= \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = \begin{vmatrix} a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \\ bc-ca & ca-ab & ab \end{vmatrix}$ $\quad C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$= \begin{vmatrix} a-b & b-c & c \\ (a+b)(a-b) & (b+c)(b-c) & c^2 \\ -c(a-b) & -a(b-c) & ab \end{vmatrix} = (a-b)(b-c)\begin{vmatrix} 1 & 1 & c \\ a+b & b+c & c^2 \\ -c & -a & ab \end{vmatrix}$

(Taking $a-b$, $b-c$ from C_1, C_2)

$= (a-b)(b-c)\begin{vmatrix} 0 & 1 & c \\ a-c & b+c & c^2 \\ a-c & -a & ab \end{vmatrix}$ $\quad C_1 \to C_1 - C_2$

$= (a-b)(b-c)(a-c)\begin{vmatrix} 0 & 1 & c \\ 1 & b+c & c^2 \\ 1 & -a & ab \end{vmatrix}$ (Taking $a-c$ from C_1)

$= -(a-b)(b-c)(c-a)\begin{vmatrix} 0 & 1 & c \\ 0 & a+b+c & c^2-ab \\ 1 & -a & ab \end{vmatrix}$ $\quad R_2 \to R_2 - R_3$

$= -(a-b)(b-c)(c-a)\,1\begin{vmatrix} 1 & c \\ a+b+c & c^2-ab \end{vmatrix}$ Expanding w.r.t C_1

$= -(a-b)(b-c)(c-a)\{c^2-ab-ac-bc-c^2\}$

$= (a-b)(b-c)(c-a)(ab+bc+ca) =$ RHS $\square$.

Ex 10: Given that the equations $x = cy+bz$, $y = az+cx$ and $z = bx+ay$ where x, y, z are not all zero. Then prove that $a^2+b^2+c^2+2abc = 1$ by determinant method.

Solⁿ: Given that $x = cy+bz$ i.e $x-cy-bz=0$ —— ①

$y = az+cx$ i.e $cx-y+az=0$ —— ②

$z = bx+ay$ i.e $bx+ay-z=0$ —— ③

Eliminating x,y,z from ①, ②, ③

$\begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0 \Rightarrow 1\begin{vmatrix} -1 & a \\ a & -1 \end{vmatrix} + c\begin{vmatrix} c & a \\ b & -1 \end{vmatrix} - b\begin{vmatrix} c & -1 \\ b & a \end{vmatrix} = 0$

$\Rightarrow 1\{(-1)(-1)-(a)(a)\} + c\{(c)(-1)-(a)(b)\} - b\{(c)(a)-(b)(-1)\} = 0$

$\Rightarrow (1-a^2) + c(-c-ab) - b(ac+b) = 0$

$\Rightarrow 1-a^2-c^2-abc-abc-b^2 = 0$

$\Rightarrow a^2+b^2+c^2+2abc = 1$ $\square$

Ex-11: Find a if $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & 2 & a-1 \end{vmatrix} = 0$

Soln: $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & 2 & a-1 \end{vmatrix} = 0$

$\Rightarrow a-1 = 3$ ($\because a_{11} = a_{31}, a_{12} = a_{32}$ and $\Delta = 0 \Rightarrow R_1$ & R_3 are identical)

$\Rightarrow a = 3+1 = 4$

Ex-12: If $A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ then find $\begin{vmatrix} 3a & 3b \\ 3c & 3d \end{vmatrix}$

Soln: $A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$\begin{vmatrix} 3a & 3b \\ 3c & 3d \end{vmatrix} = (3)(3)\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 9A$ (Ans)

Ex-13: Find the value of $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+b \end{vmatrix}$

Soln: $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+b \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ -a & a & 1 \\ 0 & -b & 1+b \end{vmatrix}$ $\quad C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$= 1\begin{vmatrix} -a & a \\ 0 & -b \end{vmatrix}$ (Expanding w.r.t R_1) $= 1(ab-0) = ab$ (Ans)

Ex-14: Let $\begin{vmatrix} 1+\alpha & \alpha & \alpha^2 \\ \alpha & 1+\alpha & \alpha^2 \\ \alpha^2 & \alpha & 1+\alpha \end{vmatrix} = a\alpha^5 + b\alpha^4 + c\alpha^3 + d\alpha^2 + e\alpha + f$ then find f, e, $a+c$, $b+d$.

Soln: Given that $\begin{vmatrix} 1+\alpha & \alpha & \alpha^2 \\ \alpha & 1+\alpha & \alpha^2 \\ \alpha^2 & \alpha & 1+\alpha \end{vmatrix} = a\alpha^5 + b\alpha^4 + c\alpha^3 + d\alpha^2 + e\alpha + f$ ——— (1)

Now $\begin{vmatrix} 1+\alpha & \alpha & \alpha^2 \\ \alpha & 1+\alpha & \alpha^2 \\ \alpha^2 & \alpha & 1+\alpha \end{vmatrix} = \begin{vmatrix} 1+2\alpha+\alpha^2 & \alpha & \alpha^2 \\ 1+2\alpha+\alpha^2 & 1+\alpha & \alpha^2 \\ 1+2\alpha+\alpha^2 & \alpha & 1+\alpha \end{vmatrix}$ $\quad C_1 \to C_1 + C_2 + C_3$

$= \begin{vmatrix} (1+\alpha)^2 & \alpha & \alpha^2 \\ (1+\alpha)^2 & 1+\alpha & \alpha^2 \\ (1+\alpha)^2 & \alpha & 1+\alpha \end{vmatrix} = (1+\alpha)^2 \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & 1+\alpha & \alpha^2 \\ 1 & \alpha & 1+\alpha \end{vmatrix}$ Taking $(1+\alpha)^2$ from C_1

$= (1+\alpha)^2 \begin{vmatrix} 0 & -1 & 0 \\ 0 & 1 & \alpha^2-\alpha-1 \\ 1 & \alpha & 1+\alpha \end{vmatrix}$ $\begin{pmatrix} R_1 \to R_1 - R_2, \\ R_2 \to R_2 - R_3 \end{pmatrix} = (1+\alpha)^2 \begin{vmatrix} -1 & 0 \\ 1 & \alpha^2-\alpha-1 \end{vmatrix}$

(Expanding w.r.t C_1)

$= (1+\alpha)^2 (\alpha+1-\alpha^2) = (1+2\alpha+\alpha^2)(\alpha+1-\alpha^2)$

$= \alpha+1-\alpha^2+2\alpha^2+2\alpha-2\alpha^3+\alpha^3+\alpha^2-\alpha^4$

$= -\alpha^4-\alpha^3+2\alpha^2+3\alpha+1$ ——— (2)

Using Eqⁿ(1) and Eq(2),

$$-\alpha^4 - \alpha^3 + 2\alpha^2 + 3\alpha + 1 = a\alpha^5 + b\alpha^4 + c\alpha^3 + d\alpha^2 + e\alpha + f \quad \text{--- (3)}$$

From (3) we have $a=0, b=-1, c=-1, d=2, e=3, f=1$

$a+c = 0+(-1) = -1$
$b+d = -1+2 = 1$

$\therefore f=1,\ e=3,\ a+c=-1,\ b+d=1.$ (Ans)

Ex 15: If B is a 3×3 matrix and $\det(3B) = R\{\det B\}$ then find R

Solⁿ: order of B $n=3$

$\det(3B) = R\{\det B\} \Rightarrow |3B| = R|B|$

$\Rightarrow 3^3|B| = R|B| \Rightarrow R = 3^3 = 27$ (Ans)

Ex 16: If x and y are square matrices of order 3, $|x|=-1$, $|y|=3$ then find $|xy|$

Solⁿ: order of $x=3$
order of $y=3$
$|x|=-1,\ |y|=3$

$|xy| = |x|\cdot|y| = (-1)(3) = -3$ (Ans)

Ex 17: Find maximum value of $\begin{vmatrix} \sin^2x & \sin x\cdot\cos x \\ -\cos x & \sin x \end{vmatrix}$

Solⁿ: Let $P(x) = \begin{vmatrix} \sin^2x & \sin x\cdot\cos x \\ -\cos x & \sin x \end{vmatrix}$

$= (\sin^2x)(\sin x) - (-\cos x)(\sin x\cdot\cos x)$

$= \sin^3x + \cos^2x\cdot\sin x = \sin x\cdot(\sin^2x+\cos^2x)$

$= \sin x\cdot 1 = \sin x.$

$\therefore \max P(x) = \max(\sin x) = 1$ (Ans)

Ex-18: Prove that $x-1$ is a factor of $\begin{vmatrix} x+1 & 2 & 3 \\ 3 & x+2 & 4 \\ 4 & 4 & x+3 \end{vmatrix}$

Solⁿ: Let $P(x) = \begin{vmatrix} x+1 & 2 & 3 \\ 3 & x+2 & 4 \\ 4 & 4 & x+3 \end{vmatrix}$

If $x-1=0 \Rightarrow x=1$

Now $P(1) = \begin{vmatrix} 1+1 & 2 & 3 \\ 3 & 1+2 & 4 \\ 4 & 4 & 1+3 \end{vmatrix} = \begin{vmatrix} 2 & 2 & 3 \\ 3 & 3 & 4 \\ 4 & 4 & 4 \end{vmatrix} = 0$ since C_1 and C_2 are identical.

$\therefore P(1) = 0 \Rightarrow x-1$ is a factor of $P(x)$.

II: MATRIX:

- A rectangular array of numbers arranged in some complete rows and in complete columns is called a matrix.
- Horizontal lines are called rows and vertical lines are called columns.
- If m = number of rows and n = number of columns of a matrix then order of the matrix is $m \times n$ or m by n.
- A matrix is denoted by capital letters of English alphabet i.e A, B, C .... also by A_1, A_2 ......
order of a matrix A is denoted by O(A) or by n(A)
-
- Elements of a matrix are included within a pair of open braket or by a pair of square braket. Elements are separated from each other by space not by comma. Elements belonging to ith row and jth column is denoted by a_{ij}. Hence a 3x2 matrix will be

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3\times2} = [a_{ij}]_{3\times2} \text{ (in compact form)}$$

- Examples of matrices are $[1]_{1\times1}$, $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times2}$,
$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times3}$, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times1}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times2}$

I: TYPES OF MATRICES:

① **RECTANGULAR MATRIX:** A matrix whose number of rows and number of columns are not equal is called rectangular matrix.

Ex: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \end{bmatrix}_{2\times3}$

② **SQUARE MATRIX:** A matrix whose number of rows and number of columns are equal is called square matrix.

Ex: $[1]_{1\times1}$, $\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}_{2\times2}$.

3: ROW MATRIX: A matrix having only one row is called row matrix.
Ex: $[2]_{1\times1}$, $[1\ \ -1\ \ 0]_{1\times3}$

4: COLUMN MATRIX: A matrix having only one column is called column matrix. Ex: $[3]_{1\times1}$, $\begin{bmatrix}1\\2\\3\end{bmatrix}_{3\times1}$

5: ZERO OR NULL MATRIX: $(O_{m\times n})$
A matrix whose all entries are zero is called zero matrix.
Ex: $[0]_{1\times1}$, $\begin{bmatrix}0&0\\0&0\end{bmatrix}_{2\times2}$, $[0\ \ 0\ \ 0]_{1\times3}$

6: HORIZONTAL MATRIX: A matrix in which number of rows is less than that of number of columns is called horizontal matrix
Ex: $\begin{bmatrix}1&0&3\\2&5&-1\end{bmatrix}_{2\times3}$

7: VERTICAL MATRIX:
A matrix in which number of rows is greater than that of number of columns is called vertical matrix.

Ex: $\begin{bmatrix}1&2\\0&1\\-1&3\end{bmatrix}_{3\times2}$

8: SINGLETON MATRIX: A matrix having only one element is called singleton matrix. Ex→ $[1]$, $[-1]$, $[0]$.

9: COMPARABLE MATRICES: Two matrices $[a_{ij}]_{m\times n}$ and $[b_{ij}]_{p\times q}$ are said to be comparable if $m=p$ & $n=q$.
Ex: $A=\begin{bmatrix}1&2&3\\-1&0&4\end{bmatrix}$, $B=\begin{bmatrix}-3&0&2\\4&-4&5\end{bmatrix}$ are comparable.

10. UNIT OR IDENTITY MATRIX:
- A square matrix whose leading diagonal elements are all unity and non-diagonal elements are all zeros is called unit matrix.
- unit matrix of order n is denoted by I_n.
- Ex: $I_1=[1]$, $I_2=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $I_3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

11. DIAGONAL MATRIX:
A square matrix whose non-diagonal elements are all zero is called diagonal matrix
Ex: $A=\begin{bmatrix}a&0&0\\0&b&0\\0&0&c\end{bmatrix}=\text{diag}[a\ \ b\ \ c]$ (Notated), $D=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

$B=\begin{bmatrix}2&0\\0&3\end{bmatrix}=\text{diag}[2\ \ 3]$, $C=\begin{bmatrix}3&0&0\\0&0&0\\0&0&6\end{bmatrix}=\text{diag}[3\ \ 0\ \ 6]$

12: SCALAR MATRIX: A diagonal matrix whose leading diagonal elements are equal and not zero is called a scalar matrix
Ex: $\begin{bmatrix}a&0&0\\0&a&0\\0&0&a\end{bmatrix}$, $(a\neq0)=[a]_{3\times3}$, $\begin{bmatrix}1&0\\0&1\end{bmatrix}=[1]_{2\times2}$, $\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}=[2]_{3\times3}$

13: UPPER TRIANGULAR MATRIX:

A square matrix in which all elements below the leading diagonal are zero is called upper triangular matrix.

Ex: $\begin{bmatrix} 3 & 2 & 5 \\ 0 & 7 & 4 \\ 0 & 0 & -2 \end{bmatrix}$

14: LOWER TRIANGULAR MATRIX:

A square matrix in which all elements above the leading diagonal are zero is called lower triangular matrix.

Ex: $\begin{bmatrix} 3 & 0 & 0 \\ -1 & 4 & 0 \\ 2 & 1 & 5 \end{bmatrix}$

15: SINGULAR MATRIX: A square matrix A is said to be singular if $\boxed{|A| = 0}$

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = (1)(6) - (2)(3) = 6 - 6 = 0 \Rightarrow A$ is singular.

16: Non-SINGULAR (Regular or invertible) matrix:

A square matrix A is said to be non-singular if $\boxed{|A| \neq 0}$

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (3)(2) = 4 - 6 = -2 \neq 0$

$\therefore$ A is non-singular.

17: SYMMETRIC MATRIX:

A square matrix A is said to be symmetric matrix if $\boxed{A^T = A}$ or $\boxed{a_{ij} = a_{ji}}$ [A^T = Transpose of A]

Ex: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = A$

$\therefore$ A is symmetric. (writing rows as columns)

18: SKEW SYMMETRIC MATRIX:

A square matrix $A = [a_{ij}]$ is said to be skew symmetric if $\boxed{A^T = -A}$ i.e $\boxed{a_{ij} = -a_{ji}}$ $\forall i, j$

Ex: $A = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$

$A^T = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix} = -A$

$\therefore$ A is skew-symmetric.

NOTE:
- The diagonal elements of a skew symmetric matrix are all zero (since $a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0 \; \forall i$) But the converse is not true.
- Every square matrix can be uniquely expressed as the sum or difference of a symmetric and a skew symmetric matrix.

$$\boxed{A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)}$$

$A+A^T \rightarrow$ symmetric
$A-A^T \rightarrow$ skew symmetric.

$$\boxed{A = \frac{1}{2}(A^T+A) - \frac{1}{2}(A^T-A)}$$

A^T, A

19: IDEMPOTENT MATRIX: A square matrix A is said to be idempotent matrix if $\boxed{A^2 = A}$

Ex: Let $A = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} \times \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} = \begin{bmatrix} 16-12 & -4+3 \\ 48-36 & -12+9 \end{bmatrix}$$ (Row by column Multiplication)

$$= \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} = A \Rightarrow A \text{ is idempotent.}$$

20: NILPOTENT MATRIX: • A square matrix A is said to be nilpotent if $\boxed{A^K = 0}$ for some positive integer K.

- The least positive value of K for which $A^K = 0$ is called index or degree or nilpotency. $K \leq n$ (order of A).
- Ex: $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$.

$$A^2 = A \times A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$$

$$= \begin{bmatrix} a^2b^2 - a^2b^2 & ab^3 - ab^3 \\ -a^3b + a^3b & -a^2b^2 + a^2b^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = (0)_{2\times 2}$$

$\therefore$ A is nilpotent of degree 2.

21: ORTHOGONAL MATRIX: A square matrix A is said to be orthogonal if $\boxed{AA^T = I}$

NOTE: • A, B are orthogonal $\Rightarrow$ AB is orthogonal.
- A is orthogonal $\Rightarrow A^T, A^{-1}$ are orthogonal
- A is orthogonal $\Rightarrow |A| = \pm 1$.
- A is orthogonal $\Rightarrow A^{-1} = A^T$

22: SINGLETON MATRIX: • A matrix having only one element is called singleton matrix
- Ex: $[a]$, $[0]$, $[-5]$ etc.

I3: INVOLUTORY MATRIX: A square matrix A is said to be involutory if $\boxed{A^2 = I}$

Ex: $A = \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}$

$$A^2 = A \times A = \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix} \times \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix} = \begin{bmatrix} (3)(3)+(2)(-4) & (3)\times 2+2\times(-3) \\ (-4)\times 3+(-3)(-4) & (-4)\times 2+(-3)(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \Rightarrow A \text{ is involutory}$$

J: SUB MATRIX: A matrix obtained from a given matrix by deleting some rows or columns or both is known as a sub matrix of the given matrix.

Ex: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ then $A_1 = [9]_{1\times 1}$, $A_2 = \begin{bmatrix} 2 & 3 \\ 5 & 6 \\ 8 & 9 \end{bmatrix}_{3\times 2}$

$A_3 = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}_{2\times 2}$, $A_4 = [8 \quad 9]_{1\times 2}$ are sub matrices of A

But $A_5 = \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix}$, $A_6 = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$ are not sub matrices of A.

J_1: PRINCIPAL SUB-MATRIX OF A SQUARE MATRIX:

A square sub-matrix 'B' of a square matrix A is said to be principal sub matrix of A if diagonal elements of B are in diagonal elements of A.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$A_1 = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$ are principal sub matrix.

Note that to obtain principal sub-matrix, delete ith. row and ith. column.

J_2: LEADING SUB-MATRIX OF A SQUARE MATRIX:

Leading sub matrix of a square matrix "A" is the square matrix which is obtained from A by deleting some of the last rows and corresponding last columns.

To obtain leading sub matrix of order K (K< order of A), delete last (n-K) rows and corresponding columns. They are always located in the top left corner of the original matrix.

Ex: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Leading sub matrix of order 2x2 is $\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$ ($\because$ Here $n=3, k=2$, $n-k = 3-2 = 1$ Delete 3rd Row & 3rd Column)

Leading sub matrix of order 1x1 is $[1]$. (Here $n-k = 3-1 = 2$ Delete last two columns and corresponding Rows)

K: TRACE OF A SQUARE MATRIX:

- Trace of a square matrix A is denoted by $Tr(A)$ and defined as the sum of all diagonal elements.
- Ex: Let $A = \begin{bmatrix} 2 & -4 & 8 \\ 0 & 3 & 5 \\ 9 & 5 & 4 \end{bmatrix}$

$Tr(A) = 2+3+4 = 9$

L: ALGEBRA OF MATRICES:

L_1: ADDITION OF TWO MATRICES: • If A and B are two matrices of same order then A and B are conformable for addition and A+B is the matrix obtained by adding elements of A with corresponding elements of B

- Mathematically if $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$ then

$A+B = [c_{ij}]_{m\times n}$ where $c_{ij} = a_{ij}+b_{ij} \ \forall \ i,j$

- Ex: $A = \begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$

$A+B = \begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 1+2 & 3+5 \\ -2+3 & 0+7 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 1 & 7 \end{bmatrix}$

- If A, B, C are three matrices of same order then
  - → $A+B = B+A$ (Commutative property)
  - → $(A+B)+C = A+(B+C)$ (Associative property)
  - → $A+O = O+A = A$ (identity property, O = identity element)
  - → $A+(-A) = (-A)+A = O$ (Inverse property, $-A$ = Additive inverse of A)
  - → $A+B = A+C \Rightarrow B = C$ (Left Cancellation Law)
    $B+A = C+A \Rightarrow B = C$ (Right Cancellation Law)

L_2: SUBSTRACTION OF TWO MATRICES: • If A and B are two matrices of same order then A and B are conformable for substraction and A−B is the matrix obtained by substracting elements of B from corresponding element of A.

- Mathematically, if $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$ then $A-B = [c_{ij}]_{m\times n}$ where $c_{ij} = a_{ij} - b_{ij} \ \forall \ i,j$
- Ex: Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}$

$A-B = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1+1 & 3-2 \\ 2-4 & 0-5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -2 & -5 \end{bmatrix}$

L_3: **SCALAR MULTIPLICATION:** If A is any matrix and K is a scalar then KA is the matrix obtained by multiplying K with each elements of A.

- Mathematically, if $A = [a_{ij}]_{m\times n}$ then $KA = [b_{ij}]_{m\times n}$ where $b_{ij} = Ka_{ij}$ $\forall$ i,j ($\forall$ stands for all)
- Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $KA = K\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} Ka & Kb \\ Kc & Kd \end{bmatrix}$

$\therefore \begin{bmatrix} Ka & Kb \\ Kc & Kd \end{bmatrix} = K\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

- If A & B are two matrices of same order, K, L are scalars then

$\rightarrow K(A+B) = KA + KB$
$\rightarrow (K+L)A = KA + LA$
$\rightarrow (KL)A = K(LA) = L(KA)$
$\rightarrow (-K)A = -(KA) = K(-A)$
$\rightarrow (1)A = A$
$\rightarrow (-1)A = -A$

L_4: **EQUALITY:**

- Two matrices A and B will be conformable for equality if they have same order and $A = B$ if their corresponding elements are equal.
- Mathematically,

$A = B \Leftrightarrow a_{ij} = b_{ij}$ $\forall$ i,j where $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$

L_5: **MATRIX MULTIPLICATION:**

- Two matrices A & B are conformable for the product AB if No. of columns of A = No of Rows of B.
- Thus if $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{n\times p}$ then A and B are conformable for the product AB of order $m\times p$ and

$AB = [c_{ij}]_{m\times p}$ where

$c_{ij} = \sum_{K=1}^{n} a_{iK} b_{Kj}, \forall i,j$ (Row by column multiplication)

In the product AB, $\underline{A}$ is called <u>pre-factor</u> (pre-multiplier) and $\underline{B}$ is called <u>post-factor</u> (post multiplier).

- Ex: Let $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$

$O(A) = 2\times 3$ and $O(B) = 3\times 2$

Hence AB is defined and $O(AB) = 2\times 2$

$$AB = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} (1)(2)+(-2)(4)+(3)(2) & (1)(3)+(-2)(5)+(3)(1) \\ (-4)(2)+(3)(4)+(5)(2) & (-4)(3)+(3)(5)+(5)(1) \end{bmatrix}$$

$= \begin{bmatrix} 0 & -4 \\ 14 & 8 \end{bmatrix}$ (Row by column product)

If A, B, C are three matrices such that conformability is assumed

- $AB \neq BA$ (not commutative)
- $A(BC) = (AB)C$ (Associative)
- $A(B+C) = AB+AC$
  $(A+B)C = AC+BC$ (Distributive)
- $AB = 0 \not\Rightarrow A=0$ or $B=0$
- $AB = AC \not\Rightarrow B=C$ even if $A \neq 0$
- $AI = IA = A$ (identity property)
- In matrix multiplication, scalar matrix behaves like a scalar multiple.

$A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ a scalar matrix

$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $AB = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (k)(a)+(0)(c) & (k)(b)+(0)(d) \\ (0)(a)+(k)(c) & (0)(b)+(k)(d) \end{bmatrix}$

$\therefore AB = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} = k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = kB.$

NOTE:
- If $AB = -BA$ then A and B are said anticommute.
- If A, B are two non zero matrices s.t $AB=0$ then A and B are called divisors of zero.
- For a square matrix A,

$$\boxed{A^n = A \cdot A \cdot A \cdot \cdots \cdot A, \text{ n times}}$$

i.e $A^2 = A\times A$, $A^3 = A\times A\times A = A^2\times A$ - - - - - -

M: TRANSPOSE OF A MATRIX:

- If A is any matrix then transpose of A is denoted by $A^T (A', \tilde{A})$ and defined by the matrix obtained by interchanging rows & columns of A
- Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times3}$ then $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3\times2}$.
- If A & B are any three matrices such that conformability is assured then
  - → $(A^T)^T = A$
  - → $(A+B)^T = A^T + B^T$
  - → $(k \cdot A)^T = k \cdot A^T$, k = scalar.
  - → $(AB)^T = B^T \cdot A^T$
  - → $(ABC)^T = C^T \cdot B^T \cdot A^T$
  - → Order of $A = m\times n \Rightarrow$ order of $A^T = n\times m$.

N: MINOR AND CO-FACTORS:

It is Exactly same that has discussed in Determinants

O: ADJOINT OF A MATRIX: (ADJUGATE):

- Adjoint of a square matrix A is denoted by Adj.A and defined by $\boxed{Adj.A = C^T}$

Where $C = [c_{ij}]_{n\times n}$ the Cofactor matrix

c_{ij} = Cofactors of a_{ij} for all i, j, $A = [a_{ij}]_{n\times n}$

- $|adj.A| = |A|^{n-1}$
- $adj.(adj.A) = |A|^{n-2}A$
- $A(adj.A) = (adj.A)A = |A| I_n$
- $adj.(KA) = K^{n-1}\, adj.A$
- $adj(A)^T = (adj.A)^T$
- $adj.(AB) = (adj.B).(adj.A)$
- $adj.(I) = I$
- $adj.(A^P) = (adj.A)^P$, $P \in Z^+$

P: INVERSE OF A SQUARE MATRIX:

- A square matrix B of order n is said to be inverse of a nonsingular square matrix A of order n if and only if $\boxed{AB = BA = I_n}$. We write it as $A^{-1} = B$.
- From definition it is clear that A^{-1} exist $\Leftrightarrow |A| \neq 0$ i.e A is non-singular.
- A square matrix A is said to be invertible if A^{-1} exist i.e $|A| \neq 0$
- $\boxed{A^{-1} = \dfrac{adj.A}{|A|}}$
- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \dfrac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- Every invertible matrix posses a unique inverse.
- $(AB)^{-1} = B^{-1}.A^{-1}$, $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$
- $(A^{-1})^{-1} = A$
- $(A^K)^{-1} = (A^{-1})^K$, $K \in N$
- $|A^{-1}| = |A|^{-1}$

Q: SOLUTION OF SYSTEM OF LINEAR EQUATION BY MATRIX METHOD.

Let's consider following system of linear equations in three variables:

$$\left.\begin{aligned} a_1x+b_1y+c_1z &= d_1 \\ a_2x+b_2y+c_2z &= d_2 \\ a_3x+b_3y+c_3z &= d_3 \end{aligned}\right\} \text{———} (1)$$

The above system of equations can be represented in matrix form as

$$AX = B$$

$$\Rightarrow A^{-1}(AX) = A^{-1}B \qquad \text{If } |A| \neq 0$$

$$\Rightarrow (A^{-1}A)X = A^{-1}B$$

$$\Rightarrow I.X = A^{-1}B \Rightarrow \boxed{X = A^{-1}B} \text{———} (2)$$

where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Q1: CONSISTENCY.

FOR NON-HOMOGENEOUS SYSTEM	FOR HOMOGENEOUS SYSTEM: $d_1=d_2=d_3=0$
i) If $\|A\| \neq 0 \Rightarrow$ system is consistent and has unique solution	i) $\|A\| \neq 0 \Rightarrow$ system is consistent and has unique trivial solution
(ii) $\|A\| = 0$, $adj(A).B = 0 \Rightarrow$ system is consistent and has infinitely many solutions.	(ii) If $\|A\| = 0 \Rightarrow$ system is consistent and has infinitely many non trivial solutions.
(iii) $\|A\| = 0$ and $adj(A).B \neq 0 \Rightarrow$ system is inconsistent. (No solutions)	(iii) No of equations < No of unknowns $\Rightarrow$ system has non trivial solution.

EX19: If $\begin{bmatrix} x-y & 2x+z \\ 2x-y & 3z+w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$ then find x, y, z, w

Soln: $\begin{bmatrix} x-y & 2x+z \\ 2x-y & 3z+w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$

$$\Rightarrow \begin{aligned} x-y &= -1 \text{ ——— } (1) \\ 2x-y &= 0 \text{ ——— } (2) \\ 2x+z &= 5 \text{ ——— } (3) \\ 3z+w &= 13 \text{ ——— } (4) \end{aligned}$$

Eqn (1) × 2 $\Rightarrow 2x-2y = -2$

Eqn (2) × 1 $\Rightarrow 2x-y = 0$

Substracting $\quad - \quad + \quad -$

$-y = -2 \Rightarrow \boxed{y = 2}$

From eqn (1), $x-y=-1 \Rightarrow x-2=-1 \Rightarrow x = -1+2 = 1 \Rightarrow \boxed{x=1}$

From eqn (3), $2x+z=5 \Rightarrow 2\times1+z=5 \Rightarrow z = 5-2 \Rightarrow \boxed{z=3}$

From equation(4), $3z+w=13 \Rightarrow 3\times3+w=13 \Rightarrow 9+w=13 \Rightarrow w=13-9$

$\Rightarrow \boxed{w=4}$

$\therefore \quad x=1, y=2, z=3, w=4$ (Ans)

Ex 20: If $A=\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}$, $B=\begin{bmatrix}6 & 5 & 4\\3 & 2 & 1\end{bmatrix}$ then find A+B.

Soln: $A=\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}$, $B=\begin{bmatrix}6 & 5 & 4\\3 & 2 & 1\end{bmatrix}$

Here order of A = order of B = 2×3

Hence A and B conformable for addition.

Now $A+B=\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}+\begin{bmatrix}6 & 5 & 4\\3 & 2 & 1\end{bmatrix}=\begin{bmatrix}1+6 & 2+5 & 3+4\\4+3 & 5+2 & 6+1\end{bmatrix}$

$=\begin{bmatrix}7 & 7 & 7\\7 & 7 & 7\end{bmatrix}$ (Ans)

Ex 21: If $A=\begin{bmatrix}1 & 2 & -3\\5 & 0 & 2\\1 & -1 & 1\end{bmatrix}$, $B=\begin{bmatrix}3 & -1 & 2\\4 & 2 & 5\\2 & 0 & 3\end{bmatrix}$ then find the matrix C such that $A+2C=B$.

Soln: $A=\begin{bmatrix}1 & 2 & -3\\5 & 0 & 2\\1 & -1 & 1\end{bmatrix}$, $B=\begin{bmatrix}3 & -1 & 2\\4 & 2 & 5\\2 & 0 & 3\end{bmatrix}$

$A+2C=B \Rightarrow 2C=B-A$.

$\Rightarrow 2C=\begin{bmatrix}3 & -1 & 2\\4 & 2 & 5\\2 & 0 & 3\end{bmatrix}-\begin{bmatrix}1 & 2 & -3\\5 & 0 & 2\\1 & -1 & 1\end{bmatrix}$

$=\begin{bmatrix}3-1 & -1-2 & 2+3\\4-5 & 2-0 & 5-2\\2-1 & 0+1 & 3-1\end{bmatrix}=\begin{bmatrix}2 & -3 & 5\\-1 & 2 & 3\\1 & 1 & 2\end{bmatrix}$

$\Rightarrow C=\frac{1}{2}\begin{bmatrix}2 & -3 & 5\\-1 & 2 & 3\\1 & 1 & 2\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\times2 & -3\times\frac{1}{2} & 5\times\frac{1}{2}\\-1\times\frac{1}{2} & 2\times\frac{1}{2} & 3\times\frac{1}{2}\\1\times\frac{1}{2} & 1\times\frac{1}{2} & 2\times\frac{1}{2}\end{bmatrix}$

$=\begin{bmatrix}1 & -\frac{3}{2} & \frac{5}{2}\\-\frac{1}{2} & 1 & \frac{3}{2}\\\frac{1}{2} & \frac{1}{2} & 1\end{bmatrix}$ (Ans)

Ex-22: Construct a 2×3 matrix having elements $a_{ij}=i+j, \forall i$

Soln: $a_{ij}=i+j, \forall i, j$ ——— (1)

Required matrix $=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\end{bmatrix}$

$=\begin{bmatrix}1+1 & 1+2 & 1+3\\2+1 & 2+2 & 2+3\end{bmatrix}$ Using Eqn(1)

$=\begin{bmatrix}2 & 3 & 4\\3 & 4 & 5\end{bmatrix}$ (Ans)

Ex23: Find B if $B^2 = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$

Soln: $B^2 = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$ —— (1)

$\Rightarrow$ order of $B = 2\times2$

Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $B^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$

$\Rightarrow a^2+bc = 17$ —— (1)

$d^2+bc = 17$ —— (2)

$ab+bd = 8$ —— (3)

$ac+cd = 8$ —— (4)

$\frac{Eq^n (3)}{Eq^n (4)} \Rightarrow \frac{ab+bd}{ac+cd} = \frac{8}{8} \Rightarrow \frac{b(a+d)}{c(a+d)} = 1 \Rightarrow \frac{b}{c} = 1 \Rightarrow b = c$ —— (5)

From (1) and (2) $\Rightarrow a^2+bc = d^2+bc \Rightarrow a^2 = d^2 \Rightarrow a = \pm d$

$\Rightarrow a = d$ —— (6) ($\because$ if $a = -d \Rightarrow ab+bd = 8 \Rightarrow -bd+bd = 8$
$\Rightarrow 0 = 8$ absured).

$a^2+bc = 17$

$\Rightarrow 1^2 + 4\times4 = 17$ or $4^2 + 1\times1 = 17$

Hence $a=1=d, b=c=4$ or $a=d=4, b=c=1$

$\therefore B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$ or $\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$ (Ans)

Ex24: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}$ then verify that $(AB)^T = B^T A^T$.

Soln: $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}$

$O(A) = 2\times3$, $O(B) = 3\times2$ Hence AB exist and $O(AB) = 2\times2$

So $O(AB)^T = 2\times2$.

Also $O(B^T) = 2\times3$, $O(A^T) = 3\times2$. Hence $B^T \cdot A^T$ exist and

$O(B^T \cdot A^T) = 2\times2$. Thus the equality is meaningful.

Now $AB = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1+4-3 & 2+0+3 \\ 3-4-1 & 6+0+1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -2 & 7 \end{bmatrix}$

$(AB)^T = \begin{bmatrix} 2 & -2 \\ 5 & 7 \end{bmatrix}$ —— (1)

$A^T = \begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 3 & 1 \end{bmatrix}$, $B^T = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix}$

$B^T \cdot A^T = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1+4-3 & 3-4-1 \\ 2+0+3 & 6+0+1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 5 & 7 \end{bmatrix}$ —— (2)

From ① & ② we have $(AB)^T = B^T A^T$

25: Find inverse of the matrix $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ if it exist and verify your Answer.

Soln: Let $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix} = 3\begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} + 4\begin{vmatrix} 2 & -3 \\ 0 & -1 \end{vmatrix}$$

$$= 3\{(-3)(1)-(-1)(4)\} + 3\{(2)(1)-(0)(4)\} + 4\{(2)(-1)-(0)(-3)\}$$

$$= 3(-3+4) + 3(2-0) + 4(-2+0) = 3+6-8 = 1 \neq 0$$

Hence A^{-1} exist

Let M_{ij} be the minor and C_{ij} be the co-factor of a_{ij}.

$C_{ij} = (-1)^{i+j} M_{ij}$ ——— ①

Using Eqn (1)

$$C_{11} = M_{11} = \begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} = -3+4 = 1$$

$$C_{12} = -M_{12} = -\begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} = -(2-0) = -2$$

$$C_{13} = M_{13} = \begin{vmatrix} 2 & -3 \\ 0 & -1 \end{vmatrix} = -2-0 = -2$$

$$C_{21} = -M_{21} = -\begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} = -(-3+4) = -1$$

$$C_{22} = M_{22} = \begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} = 3-0 = 3$$

$$C_{23} = -M_{23} = -\begin{vmatrix} 3 & -3 \\ 0 & -1 \end{vmatrix} = -(-3+0) = 3$$

$$C_{31} = M_{31} = \begin{vmatrix} -3 & 4 \\ -3 & 4 \end{vmatrix} = -12+12 = 0$$

$$C_{32} = -M_{32} = -\begin{vmatrix} 3 & 4 \\ 2 & 4 \end{vmatrix} = -(12-8) = -4$$

$$C_{33} = M_{33} = \begin{vmatrix} 3 & -3 \\ 2 & -3 \end{vmatrix} = -9+6 = -3$$

Cofactor matrix $C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

$$= \begin{bmatrix} 1 & -2 & -2 \\ -1 & 3 & 3 \\ 0 & -4 & -3 \end{bmatrix}$$

$$Adj(A) = C^T = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{Adj.(A)}{|A|} = \frac{1}{1}\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} \text{ (Ans)}$$

verification:

$$A.A^{-1} = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3+6-8 & -3-9+12 & 0+12-12 \\ 2+6-8 & -2-9+12 & 0+12-12 \\ 0+2-2 & 0-3+3 & 0+4-3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

This varifies our Answer.

Ex26: Solve following system using matrix method: $x+y+z=6$
$x-y+2z=7$, $2x-y+3z=12$

Soln: Given system is

$$\left.\begin{aligned} x+y+z&=6 \\ x-y+2z&=7 \\ 2x-y+3z&=12 \end{aligned}\right\} \text{——} (1)$$

Above system in matrix form is $AX=B \Rightarrow X=A^{-1}B$ —— (2)

where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & -1 & 3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$$

Now $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & -1 & 3 \end{vmatrix} = 1(-3+2)-1(3-4)+1(-1+2) = -1+1+1 = 1 \neq 0$

$\therefore$ A^{-1} exist. Let M_{ij} be the minor of a_{ij} and C_{ij} be the co-factor of a_{ij}. Then $C_{ij} = (-1)^{i+j}M_{ij}$ —— (3)

using Equation (3) $C_{11} = M_{11} = \begin{vmatrix} -1 & 2 \\ -1 & 3 \end{vmatrix} = -3+2 = -1$

$C_{12} = -M_{12} = -\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -(3-4) = 1$

$C_{13} = M_{13} = \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = -1+2 = 1$

$C_{21} = -M_{21} = -\begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = -(3+1) = -4$

$C_{22} = M_{22} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3-2 = 1$

$C_{23} = -M_{23} = -\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -(-1-2) = 3$

$C_{31} = M_{31} = \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = 2+1 = 3$

$C_{32} = -M_{32} = -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -(2-1) = -1$

$C_{33} = M_{33} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1-1 = -2$

Cofactor matrix $C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ -4 & 1 & 3 \\ 3 & -1 & -2 \end{bmatrix}$

$Adj.(A) = C^T = \begin{bmatrix} -1 & -4 & 3 \\ 1 & 1 & -1 \\ 1 & 3 & -2 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} \cdot Adj(A) = \frac{1}{1} \cdot \begin{bmatrix} -1 & -4 & 3 \\ 1 & 1 & -1 \\ 1 & 3 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -4 & 3 \\ 1 & 1 & -1 \\ 1 & 3 & -2 \end{bmatrix}$

From (2) $X = A^{-1}B$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & -4 & 3 \\ 1 & 1 & -1 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

$\Rightarrow x = 2, y = 1, z = 3$ (ANS)

EX-27: Solve by matrix method: $x+4y+7z=0, 2x+5y+8z=0$
$3x+6y+9z=0$

Soln: Given system is $\underset{}{x+4y+7z=0}\overset{-(1)}{,}\ 2x+5y+8z=0 \overset{-(2)}{,}\ 3x+6y+9z=0 \ -(3)$

Above system in matrix form is

$$AX = B \Rightarrow X = A^{-1}B \quad \text{—— (1)}$$

where $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = 1(45-48) - 4(18-24) + 7(12-15) = -3+24-21 = 0$

$\therefore |A| = 0$, A^{-1} doesn't exist.

$Adj.(A) \cdot B = 0 \quad \because B = 0$

Here $|A| = 0$, $Adj(A) \cdot B = 0$

$\Rightarrow$ The given system is consistent and has infinitely many solutions.

Now solving Eqn (1) & (2) for x we have

$$x = -(4y+7z) \ \& \ x = -\frac{(5y+8z)}{2}$$

$\Rightarrow -(4y+7z) = -\frac{(5y+8z)}{2}$

$\Rightarrow 8y+14z = 5y+8z \Rightarrow 3y = -6z$

$\Rightarrow \boxed{y = -2z}$, $x = -(4y+7z) = -(-8z+7z) = z$

$\Rightarrow \boxed{x = z}$

Now $x = z$ and $y = -2z$ satisfy Eqⁿ(3). for

LHS of Eqⁿ(3) $= 3\cdot z + 6(-2z) + 9z = 0 =$ RHS of Eqⁿ(3)

$\therefore\ x = z\ \ \&\ \ y = -2z$

$\Rightarrow \frac{x}{1} = z$ and $\frac{y}{-2} = z \Rightarrow \frac{x}{1} = \frac{y}{-2} = \frac{z}{1} = K$ (say)

$\Rightarrow x = K,\ y = -2K,\ z = K$ is the required solution where K is an arbitrary constant.

Ex-28: If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ then find the value of θ satisfying $A^T + A = I_2$

Solⁿ: $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},\ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

But $A^T + A = I_2 \Rightarrow \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2\cos\theta & 0 \\ 0 & 2\cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow 2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \cos\theta = \cos\frac{\pi}{3}$

$\Rightarrow \theta = 2n\pi + \frac{\pi}{3},\ n \in N$ (Ans)

Ex-29: Test the matrix $A = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$ for orthogonality.

Solⁿ: $A = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$

$A^T = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$

Now $AA^T = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4+0 & 0+0 \\ 0+0 & 0+4 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow A\cdot A^T = 4I \Rightarrow A$ is not orthogonal.

Ex-30: If the square matrix $B = [b_{ij}]_{m\times n}$ is given by $b_{ij} = (i-j)^n$ then show that B is symmetric or skew-symmetric according to n is even or odd respectively.

Solⁿ: $B = [b_{ij}]$ where $b_{ij} = (i-j)^n,\ \forall\ i,j$ ——— (1)

Now $b_{ji} = (j-i)^n = (-1)^n(i-j)^n = (-1)^n b_{ij}\ \ \forall\ i,j$ ——— (2)

If n is even then $b_{ji} = b_{ij}\ \ \forall\ i,j$ from Eqⁿ(1)

$\Rightarrow B^T = B \Rightarrow B$ is symmetric.

If n is odd then $b_{ji} = -b_{ij} \Rightarrow b_{ij} = -b_{ji}$

$\Rightarrow B^T = -B \Rightarrow B$ is skew-symmetric.

Ex-31: Express $A = \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix}$ as the sum of a symmetric and skew-symmetric matrix.

Solⁿ: $A = \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix},\ A^T = \begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix},\ A + A^T = \begin{bmatrix} 2 & 7 \\ 7 & -4 \end{bmatrix},\ A - A^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Let $P = \frac{1}{2}(A + A^T) = \frac{1}{2}\begin{bmatrix} 2 & 7 \\ 7 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 7/2 \\ 7/2 & -2 \end{bmatrix}$

$P^T = \begin{bmatrix} 1 & 7/2 \\ 7/2 & -2 \end{bmatrix} = P \Rightarrow P$ is symmetric

Let $Q = \frac{1}{2}(A - A^T) = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 \\ -1/2 & 0 \end{bmatrix}$

$$Q^T = \begin{bmatrix} 0 & -\frac{1}{2} \\ 1/2 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} = -Q$$

$\Rightarrow$ Q is skew-symmetric

But $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

$\Rightarrow \quad A = P + Q$

$\Rightarrow \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 7/2 \\ 7/2 & -2 \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}$ (Ans)

32: If the matrices A and B commute, then prove that A^{-1} and B^{-1} also commute.

Soln: A and B commute

$\Rightarrow AB = BA$

$\Rightarrow (AB)^{-1} = (BA)^{-1}$

$\Rightarrow B^{-1}.A^{-1} = A^{-1}.B^{-1} \Rightarrow A^{-1}$ and B^{-1} commute.

33. If A, B, C are three matrices conformable for multiplication then prove that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

Soln: $(ABC)^{-1} = \{A.(BC)\}^{-1} = (BC)^{-1}.A^{-1} = (C^{-1}B^{-1}).A^{-1} = C^{-1}B^{-1}A^{-1}$

34. Prove that if a matrix A is nonsingular then $AB = AC$ implies $B = C$ where B and C are square matrices of the same order as A.

Soln: A is non singular

$\Rightarrow |A| \neq 0 \Rightarrow A^{-1}$ exist

Again $AB = AC \Rightarrow A^{-1}(AB) = A^{-1}(AC)$

$\Rightarrow (A^{-1}A)B = (A^{-1}A)C$

$\Rightarrow I.B = I.C \Rightarrow B = C$

35: If $AB = AC$ then does it imply $B = C$? Can you provide a counter examples ?

Soln: If A is non-singular then $AB = AC \Rightarrow B = C$

But if A is singular then $AB = AC$ doesn't necessarily imply $B = C$.

Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $|A| = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$

$AC = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\Rightarrow A$ is singular

$\therefore AB = AC$ But $B \neq C$.

Ex-36: If A is non-singular matrix such that A is symmetric then prove that A^{-1} is also symmetric.

Soln: A is non-singular

$\Rightarrow |A| \neq 0 \Rightarrow A^{-1}$ exist

Again A is symmetric

$\Rightarrow \quad A^T = A$

$\Rightarrow \quad (A^T)^{-1} = A^{-1} \Rightarrow (A^{-1})^T = A^{-1} \Rightarrow A^{-1}$ is symmetric.

Ex-37: Show that if A is non-singular matrix then $\det(A^{-1}) = (\det A)^{-1}$

Soln: A is non-singular $\Rightarrow |A| \neq 0 \Rightarrow A^{-1}$ exist.

We know that $A \cdot A^{-1} = I \Rightarrow |AA^{-1}| = |I|$

$\Rightarrow |A| \cdot |A^{-1}| = 1 \Rightarrow |A^{-1}| = \frac{1}{|A|} \quad \because |A| \neq 0$

$\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)} \Rightarrow \det(A^{-1}) = (\det A)^{-1}$

Ex-38: If B is non singular, prove that matrices A and $B^{-1}AB$ have the same Determinant.

Soln: B is non singular $\Rightarrow |B| \neq 0 \Rightarrow B^{-1}$ exist

Now $|B^{-1}AB|$

$= |B^{-1}(AB)| = |B^{-1}| \cdot |AB| \quad \because |AB| = |A||B|$

$= |B^{-1}| \cdot |A| \cdot |B|$

$= (|B^{-1}| \cdot |B|) \cdot |A| = (|B^{-1}B|) \cdot |A| = (|I|) \cdot |A|$

$= 1 \cdot |A| = |A|$

Hence matrices A and $B^{-1}AB$ have same determinants.

Ex-39: If A is a square matrix then prove that $\operatorname{adj} A^T = (\operatorname{adj} A)^T$

Soln: $(A^T)^{-1} = \frac{\operatorname{Adj} A^T}{|A^T|} \quad \because A^{-1} = \frac{\operatorname{adj} A}{|A|}$

$\Rightarrow \operatorname{adj}(A^T) = (A^T)^{-1} \cdot |A^T|$

$\Rightarrow \operatorname{adj}(A^T) = (A^{-1})^T \cdot |A| \quad \because |A^T| = |A| \ \& \ (A^T)^{-1} = (A^{-1})^T$

$\Rightarrow \operatorname{adj}(A^T) = \left[\frac{\operatorname{adj}(A)}{|A|}\right]^T \cdot |A|$

$\Rightarrow \operatorname{adj}(A^T) = \frac{1}{|A|} \cdot (\operatorname{adj}(A))^T \cdot |A| \quad \because (KA)^T = K \cdot A^T, K$ scalar

$\Rightarrow \operatorname{adj}(A^T) = (\operatorname{adj} A)^T$ □.

Ex-40: If A is symmetric then prove that adj(A) is also symmetric.

Soln: A is symmetric $\Rightarrow A^T = A$

$\Rightarrow \operatorname{adj}(A^T) = \operatorname{adj}(A)$

$\Rightarrow [\operatorname{adj} A]^T = \operatorname{adj}(A) \Rightarrow \operatorname{adj}(A)$ is symmetric.

Ex-41: If A is both symmetric and skew symmetric then show that A is a null matrix

Soln: A is symmetric $\Rightarrow A^T = A$ —(1)

A is skew symmetric $\Rightarrow A^T = -A$ —(2)

From Eqn (1) and Eq(2), $A = -A \Rightarrow A + A = (0)$

$\Rightarrow 2A = (0) \Rightarrow A = \frac{1}{2}(0) = (0)$

$\Rightarrow$ A is a null matrix □.

Ex 42: For any 2×2 matrix D, if $D(\text{adj}\cdot D) = \begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix}$ then find $|D|$.

Soln: $D(\text{adj}\cdot D) = \begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix}$

$\Rightarrow \quad |D|\cdot I_2 = 5\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ $\qquad \because A(\text{adj} A) = |A|\cdot I_n$ where $n = O(A)$

$\Rightarrow \quad |D|\cdot I_2 = 5\, I_2$

$\Rightarrow \quad |D| = 5$ (Ans)

EXERCISE:

1) Evaluate: (i) $\begin{vmatrix}2 & 1 & 3\\ 5 & 0 & 2\\ 2 & -1 & 7\end{vmatrix}$ (ii) $\begin{vmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{vmatrix}$ (iii) $\begin{vmatrix}\sqrt5 & \sqrt{12}\\ \sqrt3 & \sqrt{20}\end{vmatrix}$

(iv) $\begin{vmatrix}a+1 & a-2\\ a+2 & a-1\end{vmatrix}$ (v) $\begin{vmatrix}\sin^2\theta & \cos^2\theta & 1\\ \cos^2\theta & \sin^2\theta & 1\\ -10 & 12 & 2\end{vmatrix}$

2) Find x if $\begin{vmatrix}1 & 5 & 7\\ 2 & x & 14\\ 3 & 1 & 2\end{vmatrix} = 0$

3) Find minors and co-factors of following determinants:

(a) $\begin{vmatrix}1 & -3 & 2\\ 4 & -1 & 2\\ 3 & 5 & 2\end{vmatrix}$ (b) $\begin{vmatrix}3 & 4\\ 7 & -2\end{vmatrix}$

4) prove following:

(a) $\begin{vmatrix}b+c & a & a\\ b & c+a & b\\ c & c & a+b\end{vmatrix} = 4abc$. (b) $\begin{vmatrix}1+a & 1 & 1\\ 1 & 1+b & 1\\ 1 & 1 & 1+c\end{vmatrix} = abc\left(\frac1a+\frac1b+\frac1c\right)$

(c) $\begin{vmatrix}x+a & b & c\\ a & x+b & c\\ a & b & x+c\end{vmatrix} = x^2(x+a+b+c)$

(d) $\begin{vmatrix}a-b-c & 2a & 2a\\ 2b & b-c-a & 2b\\ 2c & 2c & c-a-b\end{vmatrix} = (a+b+c)^3$

(e) $\begin{vmatrix}bc & a & a^2\\ ca & b & b^2\\ ab & c & c^2\end{vmatrix} = \begin{vmatrix}1 & a^2 & a^3\\ 1 & b^2 & b^3\\ 1 & c^2 & c^3\end{vmatrix}$

f.) $\begin{vmatrix}a & b & c\\ a^2 & b^2 & c^2\\ a^3 & b^3 & c^3\end{vmatrix} = abc(a-b)(b-c)(c-a)$

g. $\begin{vmatrix}x & 1 & 1\\ 1 & x & 1\\ 1 & 1 & x\end{vmatrix} = (x+2)(x-1)^2$

5. Using Cramer's Rule solve:

(a) $4x-y=9,\ 5x+2y=8$ (b) $x+2y+3z=6,\ 2x+4y+z=7,\ 3x+2y+9z=14$

(c) $2x-y=2,\ 3x+y=13$ (d) $x+y+z=3,\ 2x+3y+4z=9,\ x+2y-4z=-1$

(e) $x+2y+z=2,\ 2x-y-z=2,\ 2x-3y-2z=1$

6. Show that $a+1$ is a factor of $\begin{vmatrix} a+1 & 2 & 3 \\ 1 & a+1 & 3 \\ 3 & -6 & a+1 \end{vmatrix}$

7. Write Number of solutions of the system

(a) $x-y=0,\ 2x-2y=1$

(b) $3x+2y=1,\ x+5y=6$

(c) $x+y-z=0,\ 3x-y-z=0,\ x-3y+z=0$

8. Find x, y, z, t if $\begin{bmatrix} x+y & y-z \\ 5-t & 7+x \end{bmatrix} = \begin{bmatrix} t-x & z-t \\ z-y & x+z+t \end{bmatrix}$

9. Solve $2X-Y = \begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{bmatrix},\ 2Y+X = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$

10. If $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{bmatrix},\ B = \begin{bmatrix} 1 & 2 & -6 \\ 0 & -1 & 3 \end{bmatrix}$ then find $3A-4B$.

11) Write down the matrix $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ if $a_{ij} = 2i+3j$

12. Construct a 2×3 matrix having elements (i) $a_{ij} = i-j$ (ii) $a_{ij} = i\times j$ (iii) $a_{ij} = i/j$ $\forall\ i, j$.

(13) Find x, y if $\begin{bmatrix} 2x & y \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 8 & 3 \\ 1 & 2 \end{bmatrix}$

(14) If $A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ then find $A^3 - A^2 + I_3$

(15) Find (a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\Big[$

15) Verify that $(AB)^T = B^T A^T$ if $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix},\ B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & 3 \end{bmatrix}$

16) Show that the matrix $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ is involutory.

17) Find Adjoint of following (a) $\begin{bmatrix} -2 & 3 \\ -5 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -3 \\ -1 & 2 & 3 \end{bmatrix}$

(18) Find inverse of following:

(a) $A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$

(19) Solve following equations by matrix method:

(a) $x-y+z=4,\ 2x+y-3z=0,\ x+y+z=2$

(b) $2x-y+z=4,\ x+3y+2z=12,\ 3x+2y+3z=16$

INTEGRATION: → The reverse process of differentiation is called integration (primitive or anti-derivative). i.e If differential co-efficient of $g(x)$ w.r.t x is $f(x)$ then $g(x)$ is called integration of $f(x)$ w.r.t x.

Mathematically,

If $\frac{d}{dx} g(x) = f(x) \Rightarrow \int f(x)dx = g(x)$.

$\int f(x)dx$ is read as integral of $f(x)$ w.r.t x.

→ The function which is to be integrated is called integrand (Here $f(x)$) and the function obtained by integration is called integral. (Here $g(x)$)

→ There are two types of integration i.e Definite integral and Indefinite integral.

CONSTANT OF INTEGRATION:

→ We know that $\frac{d}{dx}(c) = 0$ where 'c' is constant.

→ Thus if $\frac{d}{dx} g(x) = f(x) \Rightarrow \frac{d}{dx}[g(x)+c] = f(x)$

$\Rightarrow \int f(x)dx = g(x)+c$. where c is an arbitrary constant known as constant of integration. Thus in General form of integration, an arbitrary constant must be added.

i.e $\int f(x)dx = g(x)+c$.

ALGEBRA OF INTEGRATION:

If $f(x)$ and $g(x)$ are two functions of x, c is a constant then

(i) $\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$.

(ii) $\int [f(x)-g(x)]dx = \int f(x)dx - \int g(x)dx$.

(iii) $\int c \cdot f(x)dx = c\int f(x)dx$

INTEGRATION OF SOME STANDARD FUNCTION:

(i) $\int dx = x+c$

(ii) $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$

(iii) $\int \frac{1}{x} dx = \log|x| + c$

(iv) $\int a^x dx = \frac{a^x}{\ln a} + c$

(v) $\int e^x dx = e^x + c$

(VI) $\int \sin x\,dx = -\cos x + c$

(VII) $\int \cos x\,dx = \sin x + c$

(VIII) $\int \tan x\,dx = \ln|\sec x| + c$

(IX) $\int \cot x\,dx = \ln|\sin x| + c$

(X) $\int \sec x\,dx = \ln|\sec x + \tan x| + c$

(XI) $\int \operatorname{cosec} x\,dx = \ln|\operatorname{cosec} x - \cot x| + c$

(XII) $\int \sec^2 x\,dx = \tan x + c$.

(XIII) $\int \operatorname{cosec}^2 x\,dx = -\cot x + c$

(XIV) $\int \sec x \cdot \tan x\,dx = \sec x + c$

(XV) $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$

(XVI) $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$

XVII) $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$

XVIII) $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1} x + c$

XIX) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

XX) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$

XXI) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c$

XXII) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln|x + \sqrt{x^2+a^2}| + c$

XXIII) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln|x + \sqrt{x^2-a^2}| + c$

XXIV) $\int \sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + c$

XXV) $\int \sqrt{a^2+x^2}\,dx = \frac{a^2}{2}\ln|x + \sqrt{a^2+x^2}| + \frac{x}{2}\sqrt{a^2+x^2} + c$

XXVI) $\int \sqrt{x^2-a^2}\,dx = -\frac{a^2}{2}\ln|x + \sqrt{x^2-a^2}| + \frac{x}{2}\sqrt{x^2-a^2} + c$

XXVII) $\int \frac{dx}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

XXVIII) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

XXIX) $\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right] + c$

XXX) $\int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right] + c$

(XXXI) If $\int f(y)\,dy = F(y) + c$

$\Rightarrow \int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + c$

Ex-1 Integrate: $\int (\sqrt{x} + \sqrt[3]{x^4} + 3x^2 + 2^x + 4)\,dx$

Sol^n: $\int (\sqrt{x} + \sqrt[3]{x^4} + 3x^2 + 2^x + 4)\,dx$

$= \int \sqrt{x}\,dx + \int \sqrt[3]{x^4}\,dx + \int 3x^2\,dx + \int 2^x\,dx + \int 4\,dx$

$= \int x^{\frac{1}{2}}\,dx + \int x^{4/3}\,dx + 3\int x^2\,dx + \int 2^x\,dx + 4\int dx$

$= \dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \dfrac{x^{\frac{4}{3}+1}}{\frac{4}{3}+1} + 3\cdot\dfrac{x^{2+1}}{2+1} + \dfrac{2^x}{\ln 2} + 4x + c$

$= \dfrac{2}{3}\sqrt{x^3} + \dfrac{3}{7}\sqrt[3]{x^7} + x^3 + \dfrac{2^x}{\ln 2} + 4x + c.$

Ex2 Evaluate: $\int \dfrac{x^4+1}{x^2+1}\,dx$

Sol^n: $\int \dfrac{x^4+1}{x^2+1}\,dx = \int \dfrac{x^4-1+2}{x^2+1}\,dx = \int \dfrac{(x^2+1)(x^2-1)+2}{(x^2+1)}\,dx$

$= \int \left(x^2 - 1 + \dfrac{2}{x^2+1}\right)dx = \int x^2\,dx - \int dx + 2\int \dfrac{dx}{1+x^2}$

$= \dfrac{1}{3}x^3 - x + 2\tan^{-1}x + c.$

Ex-3: Integrate: $\int \tan^2 x\,dx.$

Sol^n: $\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \int \sec^2 x\,dx - \int dx$

$= \tan x - x + c.$

Ex4: $\int \dfrac{1}{1+\sin x}\,dx = ?$

Sol^n: $\int \dfrac{1}{1+\sin x}\,dx = \int \dfrac{(1-\sin x)}{(1+\sin x)(1-\sin x)}\,dx = \int \dfrac{1-\sin x}{1-\sin^2 x}\,dx$

$= \int \dfrac{1-\sin x}{\cos^2 x}\,dx = \int \left(\dfrac{1}{\cos^2 x} - \dfrac{\sin x}{\cos^2 x}\right)dx = \int (\sec^2 x - \tan x \sec x)\,dx$

$= \int \sec^2 x\,dx - \int \sec x \tan x\,dx = \tan x - \sec x + c$

Ex-5: Integrate: $\int \sqrt{1+\cos 2x}\,dx$ (2017(S), 2018(S))

Sol^n: $\int \sqrt{1+\cos 2x}\,dx = \int \sqrt{2\cos^2 x}\,dx$

$= \int \sqrt{2}\cos x\,dx$

$= \sqrt{2}\sin x + c$ (Ans)

Ex-6: Integrate: $\int \dfrac{\sin x}{1+\sin x}\,dx$

Sol^n: $\int \dfrac{\sin x}{1+\sin x}\,dx = \int \dfrac{(1-\sin x)\sin x}{(1+\sin x)(1-\sin x)}\,dx = \int \dfrac{(1-\sin x)\sin x}{1-\sin^2 x}\,dx$

$= \int \dfrac{\sin x - \sin^2 x}{\cos^2 x}\,dx = \int \left(\dfrac{\sin x}{\cos^2 x} - \dfrac{\sin^2 x}{\cos^2 x}\right)dx$

$$\int\left(\frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}-\tan^2 x\right)dx=\int\left(\tan x\cdot\sec x-\sec^2 x+1\right)dx$$

$$=\int\tan x\cdot\sec x\,dx-\int\sec^2 x\,dx+\int dx$$

$$=\sec x-\tan x+x+C \quad \text{(Ans)}$$

Ex:7 Integrate: $\int\sqrt{1-\sin 2x}\,dx$

Sol^n: $\int\sqrt{1-\sin 2x}\,dx=\int\sqrt{\sin^2 x+\cos^2 x-2\sin x\cdot\cos x}\,dx$

$$=\int\sqrt{(\sin x-\cos x)^2}\,dx=\int(\sin x-\cos x)\,dx$$

$$=\int\sin x\,dx-\int\cos x\,dx=-\cos x-\sin x+c \quad \text{(Ans)}$$

Ex:8 Integrate: $\int\sin^2\left(\frac{x}{2}\right)dx$

Sol^n: $\int\sin^2\left(\frac{x}{2}\right)dx=\frac{1}{2}\int 2\sin^2\left(\frac{x}{2}\right)dx=\frac{1}{2}\int\left[1-\cos\left(2\times\frac{x}{2}\right)\right]dx$

$$=\frac{1}{2}\left[\int dx-\int\cos x\,dx\right]=\frac{1}{2}\left[x-\sin x\right]+C \quad \text{(Ans)}$$

Ex:9 Integrate: $\int\tan^{-1}(\sec x+\tan x)\,dx$

Sol^n: $\int\tan^{-1}(\sec x+\tan x)\,dx$

$$=\int\tan^{-1}\left(\frac{1}{\cos x}+\frac{\sin x}{\cos x}\right)dx$$

$$=\int\tan^{-1}\left[\frac{1+\sin x}{\cos x}\right]dx$$

$$=\int\tan^{-1}\left[\frac{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}\right]dx$$

$$=\int\tan^{-1}\left[\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}-\sin\left(\frac{x}{2}\right)\right)}\right]dx$$

$$=\int\tan^{-1}\left[\frac{\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)}{\cos\frac{x}{2}-\sin\left(\frac{x}{2}\right)}\right]dx$$

$$=\int\tan^{-1}\left[\frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\right]dx$$

Dividing N^r and D^r by $\cos\left(\frac{x}{2}\right)$

$$=\int\tan^{-1}\left[\frac{\tan\frac{\pi}{4}+\tan\left(\frac{x}{2}\right)}{1-\tan\frac{\pi}{4}\cdot\tan\frac{x}{2}}\right]dx$$

$= \int \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]dx$

$= \int\left(\frac{\pi}{4}+\frac{x}{2}\right)dx = \int\frac{\pi}{4}\,dx + \int\frac{x}{2}\,dx$

$= \frac{\pi}{4}\int dx + \frac{1}{2}\int x\,dx$

$= \frac{\pi}{4}x + \frac{1}{2}\cdot\frac{1}{2}x^2 + C$

$= \frac{\pi}{4}x + \frac{1}{4}x^2 + C$ (Ans)

Ex:10 Integrate: $\int\frac{dx}{x^2+9}$

Solⁿ: $\int\frac{dx}{x^2+9}dx = \int\frac{dx}{x^2+3^2} = \frac{1}{3}\tan^{-1}\frac{x}{3}+C$

Ex:11: Integrate: $\int\frac{dx}{\sqrt{4-x^2}}$

Solⁿ: $\int\frac{dx}{\sqrt{4-x^2}} = \int\frac{dx}{\sqrt{2^2-x^2}} = \sin^{-1}\frac{x}{2}+C$

Ex-12: Integrate: $\int\frac{dx}{x\sqrt{x^2-16}}$

Solⁿ: $\int\frac{dx}{x\sqrt{x^2-16}} = \int\frac{dx}{x\sqrt{x^2-4^2}} = \frac{1}{4}\sec^{-1}\frac{x}{4}+C$

Ex-13: Integrate: $\int\frac{dx}{\sqrt{x^2+9}}$

Solⁿ: $\int\frac{dx}{\sqrt{x^2+9}} = \int\frac{dx}{\sqrt{x^2+3^2}} = \log\left|x+\sqrt{x^2+9}\right|+C$

Ex-14: Integrate: $\int\frac{dx}{\sqrt{x^2-2}}$

Solⁿ: $\int\frac{dx}{\sqrt{x^2-2}} = \int\frac{dx}{\sqrt{x^2-(\sqrt{2})^2}} = \log\left|x+\sqrt{x^2-2}\right|+C$

Ex-15 Integrate: $\int\sqrt{4-x^2}\,dx$.

Solⁿ: $\int\sqrt{4-x^2}\,dx = \int\sqrt{2^2-x^2}\,dx = \frac{2^2}{2}\sin^{-1}\frac{x}{2} + \frac{x}{2}\sqrt{2^2-x^2}+C$

$= 2\sin^{-1}\frac{x}{2} + \frac{x}{2}\sqrt{4-x^2}+C$

Ex-16 Integrate: $\int\frac{dx}{x^2-16}$

Solⁿ: $\int\frac{dx}{x^2-16} = \int\frac{dx}{x^2-4^2} = \frac{1}{2\times4}\log\left|\frac{x-4}{x+4}\right|+C = \frac{1}{8}\log\left|\frac{x-4}{x+4}\right|+C$

EXERCISE :- 1

Evaluate following:

1: $\int\frac{x^2+2}{x^2+1}dx$ (2) $\int\cot^2x\,dx$ (3) $\int\cos^2\frac{x}{2}\,dx$ (4) $\int\sqrt{1+\sin 2x}\,dx$

(5) $\int\sqrt{1+\cos 2x}\,dx$ (6) $\int\sqrt{1-\cos 2x}\,dx$ (7) $\int\frac{dx}{1-\sin x}$ (8) $\int\frac{\sin x\,dx}{1+\sin x}$

(9) $\int\frac{dx}{1+\cos x}$ (10) $\int\frac{dx}{1-\cos x}$ (11) $\int\frac{dx}{\sec x+\tan x}$ (12) $\int\frac{dx}{(\operatorname{cosec}x-\cot x)}$

13) $\int \frac{dx}{\sec x - \tan x}$ (14) $\int \frac{dx}{\operatorname{cosec} x + \cot x}$ (15) $\int \frac{1}{\sin^2 x \cdot \cos^2 x} dx$

16) $\int \tan^{-1}\left\{\sqrt{\frac{1-\cos 2x}{1-\cos 2x}}\right\} dx$ (17) $\int \sin^{-1}(\cos x)\, dx$

18) $\int \tan^{-1}(\sec x - \tan x)\, dx$ (19) $\int e^{x+2} dx$ (20) $\int 2^x dx$

21) $\int \frac{x^2}{x^2+1} dx$ (22) $\int \left(\frac{\sec x + \tan x}{\sec x - \tan x}\right) dx$ (23) $\int e^{2\log x} \cdot dx$

23) $\int \frac{dx}{16-x^2}$ (24) $\sqrt{x^2+25}\, dx$ (25) $\int \frac{dx}{\sqrt{9-x^2}}$

F: INTEGRATION BY SUBSTITUTION AND TRIGONOMETRIC IDENTITIES:

→ When an integrand is not in standard form and it can't be converted into standard form directly, then it can be converted into standard form by suitable substitution.

→ If we substitute $\boxed{\begin{array}{l} f(x) = g(t) \\ \Rightarrow f'(x)dx = g'(t)dt \end{array}}$

→ Examples: If we put $x^2+1 = t \Rightarrow 2x\,dx = 1\cdot dt = dt$
If we put $\log x = t \Rightarrow \frac{1}{x}dx = dt$.
and so on.

→ We use following substitution in the following cases.

INTEGRAND:	SUBSTITUTION:
① $\int f(ax+b)\,dx$	put $ax+b=t$
② $\int x^{n-1} f(x^n)\,dx$	put $x^n = t$
③ $\int [f(x)]^n f'(x)\,dx$	put $f(x) = t$
④ $\int \frac{f'(x)}{f(x)}\,dx$	put $f(x) = t$
⑤ $\int \sqrt{a^2-x^2}\,dx$, $\int \frac{dx}{\sqrt{a^2-x^2}}$	put $x = a\sin\theta$
⑥ $\int \frac{1}{\sqrt{a^2+x^2}}dx$, $\int \frac{1}{a^2+x^2}dx$, $\int \sqrt{a^2+x^2}\,dx$	put $x = a\tan\theta$

INTEGRAND	SUBSTITUTION
(7) $\int\sqrt{x^2-a^2}\,dx,\ \int\frac{1}{\sqrt{x^2-a^2}}dx,\ \int\frac{1}{x^2-a^2}dx$	put $x = a\sec\theta$.
(8) $\int \sin^p x\cdot\cos^q x\,dx$	If p is odd put $\cos x = t$ If q is odd put $\sin x = t$ If p and q both are odd then put $\sin x = t,\ p>q$ $\cos x = t,\ q>p$
(9) $\int \sec^p x\cdot\tan^q x\,dx$	If p is even, put $\tan x = t$ If q is odd, put $\sec x = t$ If both p is even and q is odd, put $\sec x = t$ if $p>q$. $\tan x = t$ if $q>p$.
(10) $\int \mathrm{cosec}^p x\cdot\cot^q x\,dx$	It is same as Type-9.

9: TRIGONOMETRICAL IDENTITIES:

There are some important trigonometrical identities which are used for integration. These are given below:

(i) $\sin^2\theta+\cos^2\theta=1$ (ii) $\sec^2\theta-\tan^2\theta=1$, (iii) $\mathrm{cosec}^2\theta-\cot^2\theta=1$

(iv) $\tan\theta=\frac{\sin\theta}{\cos\theta}$ (v) $\cot\theta=\frac{\cos\theta}{\sin\theta}$ (vi) $\sin\theta\cdot\mathrm{cosec}\theta=1$

(vii) $\cos\theta\cdot\sec\theta=1$ (viii) $\tan\theta\cdot\cot\theta=1$ (ix) $\sin^{-1}(\sin x)=x$ and so on.

(x) $2\sin A\cos B=\sin(A+B)+\sin(A-B)$ (xi) $2\cos A\cdot\sin B=\sin(A+B)-\sin(A-B)$

(xii) $2\cos A\cdot\cos B=\cos(A+B)+\cos(A-B)$ (xiii) $2\sin A\cdot\sin B=\cos(A-B)-\cos(A+B)$

(xiv) $2\sin^2A=1-\cos 2A$ (xv) $2\cos^2A=1+\cos 2A$

(xvi) $\sin 3A=3\sin A-4\sin^3A$ (xvii) $\cos 3A=4\cos^3A-3\cos A$.

(xviii) $\sqrt{1+\sin 2A}=\cos A+\sin A$ (xix) $\sqrt{1-\sin 2A}=\cos A-\sin A$.

(xx) $\sin C+\sin D=2\sin\left(\frac{C+D}{2}\right)\cdot\cos\left(\frac{C-D}{2}\right)$

(xxi) $\sin C-\sin D=2\cos\left(\frac{C+D}{2}\right)\cdot\sin\left(\frac{C-D}{2}\right)$

(xxii) $\cos C+\cos D=2\cos\left(\frac{C+D}{2}\right)\cdot\cos\left(\frac{C-D}{2}\right)$

(xxiii) $\cos C-\cos D=-2\sin\left(\frac{C+D}{2}\right)\cdot\sin\left(\frac{C-D}{2}\right)$

Ex-1: Integrate: $\int\tan x\,dx$.

Sol: $\int\tan x\,dx=\int\frac{\sin x}{\cos x}dx$

$=\int-\frac{dz}{z}=-\int\frac{dz}{z}=-\ln|z|+c$

$=\ln|z^{-1}|+c=\ln\left|\frac{1}{z}\right|+c=\ln\left|\frac{1}{\cos x}\right|+c$

$=\ln|\sec x|+c$ (Ans)

put $\cos x = z$
$\Rightarrow -\sin x\,dx = dz$
$\Rightarrow \sin x\,dx = -dz$

Ex-18: Integrate: $\int \sec x\,dx$

Sol^n: $\int \sec x\,dx = \int \frac{\sec x(\sec x+\tan x)}{(\sec x+\tan x)}dx = \int \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}dx$

(put $\sec x+\tan x = t \Rightarrow (\sec x\tan x+\sec^2 x)dx = dt$)

$= \int \frac{dt}{t} = \ln|t|+c = \ln|\sec x+\tan x|+c$ (ANS)

Ex-19 Evaluate: $\int \sin(ax+b)dx$

Sol^n: $\int \sin(ax+b)dx$ — put $ax+b = z \Rightarrow a\,dx = dz \Rightarrow dx = \frac{1}{a}dz$

$= \int \sin z \cdot \frac{1}{a}dz = \frac{1}{a}\int \sin z\,dz$

$= \frac{1}{a}(-\cos z)+c = -\frac{1}{a}\cos(ax+b)$ (ANS)

Ex-20: Integrate: $\int \frac{1}{2-3x}dx$

Sol^n: $\int \frac{1}{2-3x}dx = \int \frac{-\frac{1}{3}dt}{t}$ — put $2-3x = t \Rightarrow -3dx = dt \Rightarrow dx = -\frac{1}{3}dt$

$= -\frac{1}{3}\int \frac{dt}{t} = -\frac{1}{3}\ln|t|+c$

$= -\frac{1}{3}\ln|2-3x|+c$

(21) Integrate: $\int \frac{e^x+e^{-x}}{e^x-e^{-x}}dx$

Sol^n: $\int \frac{e^x+e^{-x}}{e^x-e^{-x}}dx$ — put $e^x-e^{-x} = z \Rightarrow (e^x+e^{-x})dx = dz$

$= \int \frac{dz}{z} = \ln|z|+c = \ln|e^x-e^{-x}|+c$ (ANS)

Ex-22: Evaluate: $\int x\sin x^2 dx$

Sol^n: $\int x\sin x^2 dx$ — put $x^2 = t \Rightarrow 2x\,dx = dt \Rightarrow x\,dx = \frac{1}{2}dt$

$= \int \sin x^2\cdot(x\,dx)$

$= \int \sin t\cdot(\frac{1}{2}dt) = \frac{1}{2}\int \sin t\,dt = -\frac{1}{2}\cos t+c = -\frac{1}{2}\cos x^2+c$

Ex23: Evaluate: $\int (x-2)\sqrt{x^2-4x+7}\,dx$

Sol^n: $\int (x-2)\sqrt{(x^2-4x+7)}\,dx$

[put $x^2-4x+7 = z \Rightarrow (2x-4)dx = dz$
$\Rightarrow 2(x-2)dx = dz \Rightarrow (x-2)dx = \frac{1}{2}dz$.]

$= \frac{1}{2}\int \sqrt{z}\,dz = \frac{1}{2}\int z^{\frac{1}{2}}dz = \frac{1}{2}\cdot\frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1}+c$

$= \frac{1}{2}\times\frac{2}{3}\times z^{3/2}+c = \frac{1}{3}(x^2-4x+7)^{3/2}+c$ (Ans)

Ex24: Evaluate: $\int \frac{x^4+4x^3}{x^5+5x^4+7}dx$

Sol^n: $\int \frac{x^4+4x^3}{x^5+5x^4+7}dx$ — put $x^5+5x^4+7 = z \Rightarrow (5x^4+20x^3)dx = dz \Rightarrow 5(x^4+4x^3)dx = dz$

$= \frac{1}{5}\int \frac{5(x^4+4x^3)dx}{(x^5+5x^4+7)}$

$= \frac{1}{5}\int \frac{dz}{z} = \frac{1}{5}\ln|z|+c = \frac{1}{5}\ln|x^5+5x^4+7|+c$ (Ans)

Ex21: Integrate: $\int(3x+5)^7 dx$.

soln: $\int(3x+5)^7.dx = \int u^7.\frac{1}{3}du$ $\quad\left[\begin{array}{l}\text{put } 3x+5=u\\ \Rightarrow 3dx=du\\ \Rightarrow dx=\frac{1}{3}du\end{array}\right.$

$= \frac{1}{3}\int u^7.du = \frac{1}{3}\times\frac{1}{8}u^8+c$

$= \frac{1}{24}(3x+5)^8+c$ (Ans)

Ex22: Evaluate: $\int\frac{1}{e^x-1}dx$

soln: $\int\frac{1}{e^x-1}dx = \int\frac{e^{-x}}{e^{-x}(e^x-1)}dx = \int\frac{e^{-x}dx}{(1-e^{-x})}$

[put $1-e^{-x}=t \Rightarrow [0-(-e^{-x})]dx = dt \Rightarrow e^{-x}dx = dt$]

$= \int\frac{dt}{t} = \ln|t|+c = \ln|1-e^{-x}|+c$

Ex23: Evaluate: $\int\frac{1}{1+\tan x}dx$.

soln: $\int\frac{1}{1+\tan x}dx = \int\frac{1}{1+\frac{\sin x}{\cos x}}dx = \int\frac{\cos x}{\cos x+\sin x}dx$

$= \frac{1}{2}\int\frac{2\cos x}{\cos x+\sin x}dx = \frac{1}{2}\int\frac{(\cos x+\sin x)+(\cos x-\sin x)}{(\cos x+\sin x)}dx$

$= \frac{1}{2}\int\left(1+\frac{\cos x-\sin x}{\cos x+\sin x}\right)dx = \frac{1}{2}\int dx+\frac{1}{2}\int\frac{\cos x-\sin x}{\cos x+\sin x}dx$

$= \frac{1}{2}x+\frac{1}{2}\int\frac{dt}{t}$

put $\cos x+\sin x = t \Rightarrow (-\sin x+\cos x)dx = dt$
$\Rightarrow (\cos x-\sin x)dx = dt$.

$= \frac{1}{2}x+\frac{1}{2}\ln|t|+c = \frac{1}{2}x+\frac{1}{2}\ln|\cos x+\sin x|+c$.

$= \frac{1}{2}[x+\ln|\cos x+\sin x|]+c$ (Ans)

Ex25: Evaluate: $\int\sin^4 x\,dx$

soln: $\int\sin^4 x\,dx = \frac{1}{4}\int(2\sin^2 x)^2dx = \frac{1}{4}\int(1-\cos 2x)^2dx = \frac{1}{4}\int(1+\cos^2 2x-2\cos 2x)dx$

$= \frac{1}{4}\left\{\int dx+\int\cos^2 2x\,dx-2\int\cos 2x\,dx\right\}$

$= \frac{1}{4}\left\{x+\frac{1}{2}\int 2\cos^2 2x\,dx-2\times\frac{1}{2}\sin 2x\right\}$

$= \frac{1}{4}\left\{x+\frac{1}{2}\int(1+\cos 4x)dx-\sin 2x\right\}$

$= \frac{1}{4}x+\frac{1}{8}\int dx+\frac{1}{8}\int\cos 4x\,dx-\frac{1}{4}\sin 2x$

$= \frac{1}{4}x+\frac{1}{8}x+\frac{1}{8}\times\frac{1}{4}\sin 4x-\frac{1}{4}\sin 2x+c$

$= \frac{3}{8}x+\frac{1}{32}\sin 4x-\frac{1}{4}\sin 2x+c$ (ANS)

Ex29: Evaluate: $\int\cos^3 x\,dx$.

soln: $\int\cos^3 x.dx = \int\cos x.\cos^2 x\,dx = \int\cos x(1-\sin^2 x)dx$

[put $\sin x = t \Rightarrow \cos x\,dx = dt$]

$= \int(1-t^2)dt = \int dt-\int t^2dt = t-\frac{1}{3}t^3+c$

$= \sin x-\frac{1}{3}\sin^3 x+c$ (Ans)

Ex 30: $\int \cos x \cos 2x \cos 3x \, dx$

Soln: $\int \cos x \cdot \cos 2x \cdot \cos 3x \, dx = \frac{1}{2}\int (2\cos 2x \cos x)\cos 3x \, dx$

$= \frac{1}{2}\int [\cos(2x+x) + \cos(2x-x)]\cos 3x \, dx = \frac{1}{2}\int (\cos^2 3x + \cos 3x \cdot \cos x)\,dx$

$= \frac{1}{2}\int \cos^2 3x \, dx + \frac{1}{2}\int \cos 3x \cdot \cos x \, dx = \frac{1}{4}\int 2\cos^2 3x \, dx + \frac{1}{4}\int 2\cos 3x \cos x \, dx$

$= \frac{1}{4}\int (1+\cos 6x)\,dx + \frac{1}{4}\int (\cos 4x + \cos 2x)\,dx$

$= \frac{1}{4}\int dx + \frac{1}{4}\int \cos 6x \, dx + \frac{1}{4}\int \cos 4x \, dx + \frac{1}{4}\int \cos 2x \, dx$

$= \frac{1}{4}x + \frac{1}{4}\times\frac{1}{6}\sin 6x + \frac{1}{4}\times\frac{1}{4}\sin 4x + \frac{1}{4}\times\frac{1}{2}\sin 2x + C$

$= \frac{1}{4}x + \frac{1}{24}\sin 6x + \frac{1}{16}\sin 4x + \frac{1}{8}\sin 2x + C$ (ANS)

Ex 31: $\int \sin^3 x \cdot \sec^{11} x \, dx$

Soln: $\int \sin^3 x \cdot \sec^{11} x \, dx = \int \frac{\sin^3 x}{\cos^3 x} \cdot \sec^{11} x \, dx$

$= \int \tan^3 x \cdot \sec^{11} x \, dx = \int \tan^2 x \cdot \sec^{10} x \cdot (\sec x \cdot \tan x \, dx)$

$= \int (\sec^2 x - 1)\sec^{10} x \cdot (\sec x \tan x \, dx)$

[put $\sec x = z \Rightarrow \sec x \tan x \, dx = dt$]

$= \int (z^2-1) z^{10} dz = \int z^{12} dz - \int z^{10} dz = \frac{1}{13}z^{13} - \frac{1}{11}z^{11} + C$

$= \frac{1}{13}\sec^{13} x - \frac{1}{11}\sec^{11} x + C$ (ANS)

Ex-32: $\int \frac{dx}{x^2+6x+13}$

Soln: $\int \frac{dx}{x^2+6x+13} = \int \frac{dx}{(x)^2 + 2(x)(3) + (3)^2 - (3)^2 + 13}$

$= \int \frac{dx}{(x+3)^2 + 4} = \int \frac{dt}{t^2+2^2}$ [put $x+3 = t \Rightarrow dx = dt$]

$= \frac{1}{2}\cdot\tan^{-1}\left(\frac{t}{2}\right) + C = \frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right) + C$ (ANS)

Ex: 33: $\int \frac{dx}{x \ln x \sqrt{(\ln x)^2 - 4}}$

Soln: $\int \frac{dx}{x \ln x \sqrt{(\ln x)^2-4}} = \int \frac{1}{\ln x \sqrt{(\ln x)^2 - 4}} \cdot \left(\frac{1}{x}dx\right)$

[put $\ln x = t \Rightarrow \frac{1}{x}dx = dt$]

$= \int \frac{1}{t\sqrt{t^2-2^2}}dt = \frac{1}{2}\sec^{-1}\left(\frac{t}{2}\right) + C = \frac{1}{2}\sec^{-1}\left(\frac{\ln x}{2}\right) + C$ (ANS)

Ex 34: Integrate: $\int \frac{e^{4x}}{e^{8x}+4}dx$

Soln: $\int \frac{e^{4x}}{e^{8x}+4}dx = \frac{1}{4}\int \frac{4e^{4x}\,dx}{(e^{4x})^2+4}$ put $e^{4x} = t \Rightarrow 4\cdot e^{4x}dx = dt$

$= \frac{1}{4}\int \frac{dt}{t^2+2^2} = \frac{1}{4}\times\frac{1}{2}\times\tan^{-1}\left(\frac{t}{2}\right) + C$

$= \frac{1}{8}\tan^{-1}\left(\frac{e^{4x}}{2}\right) + C$ (ANS)

EXERCISE

Integrate following:

(1) $\int \cot x \, dx$ (2) $\int \operatorname{cosec} x \, dx$ (3) $\int (4x+1)^{100} dx$ (4) $\int \sin(5x+2) dx$

(5) $\int \frac{\log x}{x} dx$ (6) $\int \frac{\sec^2(\log x)}{x} dx$ (7) $\int \frac{e^{\tan^{-1}x}}{1+x^2} dx$.

(8) $\int \cos(ax+b) dx$ (9) $\int \tan(2x+b) dx$ (10) $\int \frac{dx}{2x+7}$ (11) $\int \frac{dx}{5-3x}$

(12) $\int \sqrt{\sin x} \cdot \cos x \, dx$ (13) $\int \frac{3x^2}{1+x^6} dx$ (14) $\int \frac{4x+3}{\sqrt{2x^2+3x+1}} dx$ (15) $\int \frac{dx}{\sqrt{x}+x}$

(16) $\int \frac{\cos x - \sin x}{\cos x + \sin x} dx$ (17) $\int \frac{\sec x}{\log(\sec x + \tan x)} dx$ (18) $\int \frac{\sin x \cdot \cos x}{a^2 \sin^2 x + b^2 \cos^2 x} dx$

(19) $\int \frac{x^2 \tan^{-1} x^3}{1+x^6} dx$ (20) $\int \frac{dx}{1+e^{-x}}$ (21) $\int \frac{7}{2-3x} dx$ (22) $\int \frac{dx}{x[1+(\ln x)^2]}$

(23) $\int \frac{(x+1) \ln(x^2+2x+2)}{(x^2+2x+2)} dx$ (24) $\int \sin^3 x \, dx$ (25) $\int \sin^4 x \, dx$

(26) $\int \sin^2 x \, dx$ (27) $\int \cos^2 x \, dx$ (28) $\int \cos^3 x \, dx$ (29) $\int \cos^2 mx \, dx$

(30) $\int \sin 3x \cdot \sin 2x \, dx$ (31) $\int \cos 4x \cdot \cos x \, dx$ (32) $\int \cos 3x \cdot \sin 2x \, dx$

(33) $\int \sin^3 x \cos^3 x \, dx$ (34) $\int \sin^{12} x \cos^{13} x \, dx$ (35) $\int \sec^4 x \cdot \tan x \, dx$

(36) $\int \cot^3 x \operatorname{cosec}^{16} x \, dx$ (37) $\int \tan^5 \theta \cdot \sec^4 \theta \, d\theta$

(38) $\int \sin 3x \cdot \cos^3 x \, dx$ (39) $\int \cot^4 \theta \, d\theta$ (40) $\int \frac{\sin ax - \sin bx}{\cos ax - \cos bx} dx$

(41) $\int \frac{\sin 4x - \sin 2x}{\cos x} dx$ (42) [illegible]

H: INTEGRATION BY PARTS:

→ If the integrand is expressed as product of two functions of same variable and it can't be change into any standard form then we apply the method known as integration by parts.

→ If p and q are two functions of x which are differentiable then,

$$\frac{d}{dx}(pq) = p\cdot\frac{dq}{dx} + q\frac{dp}{dx}$$

$$\Rightarrow p\frac{dq}{dx} = \frac{d}{dx}(pq) - q\cdot\frac{dp}{dx}$$

$$\Rightarrow \int p\frac{dq}{dx}\cdot dx = \int \frac{d}{dx}(pq)\cdot dx - \int q\cdot\frac{dp}{dx}\cdot dx$$

$$\Rightarrow \int p\frac{dq}{dx}\cdot dx = pq - \int q\cdot\frac{dp}{dx}\cdot dx$$

$$\left(\text{Put } p = u,\ v = \frac{dq}{dx} \Rightarrow \int v\,dx = q\right)$$

$$\Rightarrow \boxed{\int uv\cdot dx = u\int v\,dx - \int\left(\int v\,dx\right)\cdot\frac{d}{dx}(u)\cdot dx}$$

This Rule is called Integration by parts and used to integrate product of two function.

Here u is called 1st function and v is called 2nd. function

→ The choice of 1st function is made basin on order of "ILATE" where

I — Inverse Trigonometric function
L = Logarithm function
A = Algebraic function
T = Trigonometric function
E = Exponential function.

Example: $\int xe^x dx$ take x as 1st function (Algebraic) and e^x as 2nd function (exponential function)

→ But "ILATE" rule is not applicable always. In that case the function whose integral is known, is taken as 2nd function.

→ If integral of both functions are known, then the function whose derivative vanishes rapidly can be taken as 1st function.

→ When the integrand contains only one function then 1 can be taken as other function

→ When we need to apply integration by parts successively we use following integration by parts formula.

$$\boxed{\begin{array}{l}\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots\cdots \\ \text{where } u' = \frac{du}{dx},\ u'' = \frac{du'}{dx} \cdots\cdots \\ v_1 = \int v\,dx,\ v_2 = \int v_1\,dx \cdots\cdots\end{array}}$$

SPECIAL FORM:

To integrate $\int e^x[f(x)+f'(x)]dx = \int e^x f(x)dx + \int e^x f'(x)dx$

$\downarrow$ Apply by parts taking $f(x)$ as 1st function

$\downarrow$ Keep it as it is

Ex 35: Integrate: $\int x\cos x\,dx$. $= e^x f(x)$.

Soln: $\int x\cos x\,dx = x\int \cos x\,dx - \int\left(\int \cos x\,dx)\times \frac{d}{dx}x\right)\cdot dx$ (Integrating by parts)

$= x\sin x - \int \sin x\cdot 1\cdot dx$

$= x\sin x - (-\cos x)+c = x\sin x + \cos x + c$ (Ans)

36: Integrate: $\int \ln x\,dx$

Soln: $\int \ln x\,dx = \int 1\cdot \ln x\,dx = \ln x\int 1\cdot dx - \int\left(\int 1\cdot dx)\times \frac{d}{dx}\ln x\right)dx$

$= \ln x\cdot x - \int x\cdot \frac{1}{x}\cdot dx = x\ln x - \int dx$

$= x\ln x - x + c$ (Ans)

Ex 37 :- Integrate : $\int x^2 e^x dx$

Soln : $\int x^2 e^x dx$

$= [x^2](e^x) - (2x)(e^x) + (2)(e^x) - (0)\,e^x + c$

$\left[\because \frac{d}{dx}x^2 = 2x,\ \frac{d}{dx}(2x) = 2,\ \frac{d}{dx}(2) = 0 \right.$
$\left. \text{and } \int e^x dx = e^x, \text{ Applying formula (iv)}\right]$

$= x^2 e^x - 2xe^x + 2e^x + c$

$= (x^2 - 2x + 2)\,e^x + c$

38:- Integrate : $\int (x^2+1)\sin x\,dx$

Soln $\int (x^2+1)\sin x\,dx$

$= (x^2+1)[-\cos x] - (2x)(-\sin x) + (2)[\cos x] + c$

$\left[\because \frac{d}{dx}(x^2+1) = 2x,\ \frac{d}{dx}2x = 2,\ \frac{d}{dx}(2) = 0\right.$

$\int \sin x\,dx = -\cos x,\ \int -\cos x\,dx = -\sin x,$

$\left.\int -\sin x\,dx = \cos x \text{ and using llegheroster by parts}\right]$

$= -(x^2+1)\cos x + 2x\sin x + 2\cos x + c$ (Ans)

39:- Integrate: $\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$

Soln $\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$

$$\left[\text{Here } \frac{d}{dx}\left(\frac{1}{x}\right)=\frac{d}{dx}(x^{-1})=(-1)\,x^{-1-1}=-x^{-2}=-\frac{1}{x^2}\right]$$

Here $f(x)=\frac{1}{x}$, $f'(x)=-\frac{1}{x^2}$,

$$=\int e^x\frac{1}{x}dx-\int e^x\frac{1}{x^2}dx$$

$$=\left[\frac{1}{x}\int e^x dx-\int\left(\int e^x dx\right)\times\frac{d}{dx}\left(\frac{1}{x}\right)dx\right]-\int\frac{e^x}{x^2}dx$$

(Integrating by parts taking $\frac{1}{x}$ as Ist function)

$$=\frac{1}{x}e^x-\int e^x\left(-\frac{1}{x^2}\right)dx-\int\frac{e^x}{x^2}dx+c$$

$$=\frac{1}{x}e^x+\cancel{\int\frac{e^x}{x^2}dx}-\cancel{\frac{e^x}{x^2}dx}+c$$

$$=\frac{1}{x}e^x+c \quad \text{(Ans)}$$

40:- Integrate: $\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx$

Soln $\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx$

$$=\int e^x\left(\frac{1}{1+\cos x}+\frac{\sin x}{1+\cos x}\right)dx$$

$$\left[\text{Here } \frac{d}{dx}\left(\frac{\sin x}{1+\cos x}\right)=\frac{(1+\cos x)\cos x-\sin x(-\sin x)}{(1+\cos x)^2}\right.$$

$$\left.=\frac{\cos x+\cos^2 x+\sin^2 x}{(1+\cos x)^2}=\frac{1+\cos x}{(1+\cos x)^2}=\frac{1}{(1+\cos x)}\right]$$

$$=\int e^x\left[\frac{\sin x}{1+\cos x}+\frac{1}{1+\cos x}\right]dx$$

$$=\int e^x\frac{\sin x}{1+\cos x}dx+\int e^x\frac{1}{1+\cos x}dx$$

$$= \left[\frac{\sin x}{1+\cos x}\int e^x dx - \int\left(\int e^x dx\right)\times\frac{d}{dx}\left(\frac{\sin x}{1+\cos x}\right)dx\right] + \int e^x \frac{1}{1+\cos x}dx$$

(Integrating by parts)

$$= \frac{\sin x}{1+\cos x}\cdot e^x - \int e^x \cdot \frac{1}{1+\cos x}dx + \int e^x \frac{1}{1+\cos x}dx + c$$

$$= \frac{e^x \sin x}{1+\cos x} + c \quad \text{(Ans)}$$

:- EXERCISE :-

Integrate

1. $\int \tan^{-1}x \, dx$
2. $\int x^2 e^{3x} dx$
3. $\int x^2 \cos^2 x \, dx$
4. $\int (\log x)^2 dx$
5. $\int \log\left(x+\sqrt{x^2+a^2}\right) dx$
6. $\int \left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right] dx$
7. $\int \frac{xe^x}{(x+1)^2} dx$
8. $\int \frac{\log x}{(1+\log x)^2} dx$
9. $\int x \tan^{-1}x \, dx$
10. $\int x^2 \sin^{-1}x \, dx$
11. $\int \log(1+x^2) \, dx$
12. $\int e^x (\sin x + \cos x) \, dx$
13. $\int \frac{e^x (1 + x\log x)}{x} dx$
14. $\int e^x \left(\frac{1}{x^2} - \frac{2}{x^3}\right) dx$
15. $\int e^x \left(\cot x + \log|\sin x|\right) dx$

Scanned with CamScanner

I: INTEGRAL OF FOLLOWING FORMS:

i) $\int \frac{dx}{x^2+a^2}$ (ii) $\int \frac{dx}{x^2-a^2}$ (iii) $\int \frac{dx}{a^2-x^2}$ (iv) $\int \frac{dx}{\sqrt{x^2+a^2}}$ (v) $\int \frac{dx}{\sqrt{x^2-a^2}}$

(vi) $\int \frac{dx}{\sqrt{a^2-x^2}}$ (vii) $\int \frac{dx}{x\sqrt{x^2-a^2}}$ (viii) $\int \sqrt{a^2-x^2}\,dx$ (IX) $\int \sqrt{a^2+x^2}\,dx$

(X) $\int \sqrt{x^2-a^2}\,dx$ (xi) $\int e^{ax}\sin bx\,dx$ (xii) $\int e^{ax}\cos bx\,dx$.

Ex-41: Integrate : $\int \frac{dx}{x^2+a^2}$ (General form(i))

Soln: $\int \frac{dx}{x^2+a^2}$

put $x = a\tan\theta$
$\Rightarrow dx = a\sec^2\theta \cdot d\theta$
& $\theta = \tan^{-1}\left(\frac{x}{a}\right)$

$= \int \frac{a\sec^2\theta \cdot d\theta}{a^2\tan^2\theta + a^2}$

$= \int \frac{a\sec^2\theta\, d\theta}{a^2(\tan^2\theta+1)} = \int \frac{a\sec^2\theta}{a^2\sec^2\theta}d\theta = \frac{1}{a}\int d\theta = \frac{1}{a}\theta = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

$\therefore \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$ (Ans.)

Ex42: Integrate : $\int \frac{dx}{x^2-25}$ (particular form ii)

Soln: $\int \frac{dx}{x^2-25} = \int \frac{dx}{(x)^2-(5)^2} = \int \frac{dx}{(x+5)(x-5)} = \frac{1}{10}\int\left(\frac{1}{x-5} - \frac{1}{x+5}\right)dx$

$= \frac{1}{10}\left\{\int \frac{dx}{x-5} - \int \frac{dx}{x+5}\right\} =$

put $x-5 = t \Rightarrow dx = dt$
and $x+5 = z \Rightarrow dx = dz$

$= \frac{1}{10}\left\{\frac{dt}{t} - \frac{dz}{z}\right\} = \frac{1}{10}\left\{\ln|t| - \ln|z|\right\} + c$

$= \frac{1}{10}\left\{\ln\left|\frac{t}{z}\right|\right\} + c = \frac{1}{10}\ln\left|\frac{x-5}{x+5}\right| + c$ (ANS)

Ex43: Integrate : $\int \frac{dx}{a^2-x^2}$

Soln: $\int \frac{dx}{a^2-x^2} = \int \frac{dx}{(a+x)(a-x)} = \frac{1}{2a}\int\left(\frac{1}{a+x} + \frac{1}{a-x}\right)dx$

$= \frac{1}{2a}\left\{\int \frac{dx}{a+x} + \int \frac{dx}{a-x}\right\} = \frac{1}{2a}\left\{\int \frac{dt}{t} + \int -\frac{dz}{z}\right\}$

(put $a+x = t \Rightarrow dx = dt$
& $a-x = z \Rightarrow -dx = dz \Rightarrow dx = -dz$)

$= \frac{1}{2a}\left\{\ln|t| - \ln|z|\right\} + c = \frac{1}{2a}\left\{\ln\left|\frac{t}{z}\right|\right\} + c$

$= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$

Ex: Integrate: $\int \frac{dx}{\sqrt{x^2+25}}$

Solⁿ: $\int \frac{dx}{\sqrt{x^2+25}} = \int \frac{dx}{\sqrt{x^2+5^2}} = \int \frac{5\sec^2\theta d\theta}{\sqrt{25\tan^2\theta+25}}$ put $x = 5\tan\theta$ $\Rightarrow dx = 5\sec^2\theta d\theta$

$= \int \frac{5\sec^2\theta d\theta}{5\sqrt{\tan^2\theta+1}} = \int \frac{\sec^2\theta d\theta}{\sqrt{\sec^2\theta}} = \int \frac{\sec^2\theta d\theta}{\sec\theta} = \int \sec\theta d\theta$

$= \ln|\sec\theta+\tan\theta| + c_1$

$\left(x = 5\tan\theta \Rightarrow \tan\theta = \frac{x}{5}, \sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\frac{x^2}{25}} = \frac{\sqrt{x^2+25}}{5}\right)$

$= \ln\left|\frac{\sqrt{x^2+25}}{5} + \frac{x}{5}\right| + c_1 = \ln\left|\frac{x+\sqrt{x^2+25}}{5}\right| + c_1$

$= \ln|x+\sqrt{x^2+25}| - \ln 5 + c_1 = \ln|x+\sqrt{x^2+25}| + c$ (Ans)

where $c = c_1 - \ln 5$

45) Integrate: $\int \frac{dx}{\sqrt{x^2-a^2}}$

Solⁿ: $\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sec\theta.\tan\theta.d\theta}{\sqrt{a^2\sec^2\theta-a^2}} = \int \frac{a\sec\theta.\tan\theta.d\theta}{a\sqrt{\sec^2\theta-1}}$ put $x = a\sec\theta$ $dx = a\sec\theta.\tan\theta d\theta$

$= \int \frac{a\sec\theta.\tan\theta.d\theta}{a\sqrt{\tan^2\theta}} = \int \frac{a\sec\theta.\tan\theta}{a\tan\theta}.d\theta = \int \sec\theta d\theta = \ln|\sec\theta+\tan\theta| + c_1$

$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + c_1 = \ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + c_1 = \ln|x+\sqrt{x^2-a^2}| - \ln a + c_1$

$\left(x = a\sec\theta \Rightarrow \sec\theta = \frac{x}{a}, \tan\theta = \sqrt{\sec^2\theta-1} = \sqrt{\frac{x^2}{a^2}-1} = \frac{\sqrt{x^2-a^2}}{a}\right)$

$= \ln|x+\sqrt{x^2-a^2}| + c$ (Taking $c = c_1 - \ln a$) (Ans)

Ex-46: Integrate: $\int \frac{dx}{\sqrt{a^2-x^2}}$

Solⁿ: $\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta d\theta}{\sqrt{a^2-a^2\sin^2\theta}}$ put $x = a\sin\theta$ $\Rightarrow dx = a\cos\theta d\theta$

$= \int \frac{a\cos\theta.d\theta}{\sqrt{a^2(1-\sin^2\theta)}} = \int \frac{a\cos\theta.d\theta}{a\sqrt{\cos^2\theta}} = \int \frac{a\cos\theta}{a\cos\theta} d\theta = \int d\theta$

$= \theta + c = \sin^{-1}\frac{x}{a} + c$ $\because \sin\theta = \frac{x}{a} \Rightarrow \theta = \sin^{-1}\frac{x}{a}$.

Ex-47: Integrate: $\int \frac{dx}{x\sqrt{x^2-a^2}}$.

Solⁿ: $\int \frac{dx}{x\sqrt{x^2-a^2}} = \int \frac{a\sec\theta.\tan\theta.d\theta}{a\sec\theta.\sqrt{a^2\sec^2\theta-a^2}}$ put $x = a\sec\theta$ $dx = a\sec\theta.\tan\theta d\theta$

$= \int \frac{a\sec\theta.\tan\theta.d\theta}{a\sec\theta\sqrt{a^2(\sec^2\theta-1)}} = \int \frac{\tan\theta.d\theta}{a\sqrt{\sec^2\theta-1}} = \int \frac{\tan\theta.d\theta}{a\sqrt{\tan^2\theta}} = \int \frac{\tan\theta\, d\theta}{a\tan\theta}$

$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta + c = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c$ (Ans)

$\because x = a\sec\theta \Rightarrow \frac{x}{a} = \sec\theta \Rightarrow \theta = \sec^{-1}\frac{x}{a}$.

Ex-48: Integrate: $\int\sqrt{a^2-x^2}\,dx$.

Soln: $\int\sqrt{a^2-x^2}\,dx = \int\sqrt{a^2-a^2\sin^2\theta}\cdot a\cos\theta\,d\theta$ put $x=a\sin\theta$ $\Rightarrow dx = a\cos\theta\,d\theta$

$= \int a\sqrt{1-\sin^2\theta}\cdot a\cos\theta\,d\theta = a^2\int\cos^2\theta\,d\theta = \frac{a^2}{2}\int 2\cos^2\theta\,d\theta$

$= \frac{a^2}{2}\int(1+\cos 2\theta)\,d\theta = \frac{a^2}{2}\int d\theta + \frac{a^2}{2}\int\cos 2\theta\,d\theta = \frac{a^2}{2}\theta + \frac{a^2}{4}\sin 2\theta + c$

$= \frac{a^2}{2}\theta + \frac{a^2}{4}\cdot 2\sin\theta\cdot\cos\theta + c = \frac{a^2}{2}\theta + \frac{a^2}{2}\cdot\sin\theta\cdot\cos\theta + c$

$= \frac{a^2}{2}\cdot\sin^{-1}\left(\frac{x}{a}\right) + \frac{a^2}{2}\cdot\frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a} + c$ $\because \sin\theta = \frac{x}{a}$

$\cos\theta = \sqrt{1-\sin^2\theta}$

$= \sqrt{1-\frac{x^2}{a^2}}$

$= \frac{\sqrt{a^2-x^2}}{a}$

$= \frac{a^2}{2}\cdot\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + c$

Ex-49: Integrate: $\int\sqrt{a^2+x^2}\,dx$.

Soln: Let, $I = \int\sqrt{a^2+x^2}\,dx$

$= \sqrt{a^2+x^2}\int dx - \int\left(\int dx\right)\times\left(\frac{d}{dx}\sqrt{a^2+x^2}\right)dx$ (Integrating by parts)

$= \sqrt{a^2+x^2}\cdot x - \int x\cdot\frac{1}{2\sqrt{a^2+x^2}}\cdot 2x\cdot dx$

$= x\cdot\sqrt{a^2+x^2} - \int\frac{x^2}{\sqrt{a^2+x^2}}\,dx = x\cdot\sqrt{a^2+x^2} - \int\frac{x^2+a^2-a^2}{\sqrt{a^2+x^2}}\,dx$

$= x\sqrt{a^2+x^2} - \int\sqrt{a^2+x^2}\,dx + a^2\int\frac{dx}{\sqrt{a^2+x^2}}$

$= x\sqrt{a^2+x^2} - I + a^2\ln\left|x+\sqrt{a^2+x^2}\right| + c_1$

$\Rightarrow 2I = a^2\ln\left|x+\sqrt{a^2+x^2}\right| + x\sqrt{a^2+x^2} + c_1$

$\Rightarrow I = \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right| + \frac{x}{2}\sqrt{a^2+x^2} + \frac{c_1}{2}$

$\Rightarrow \int\sqrt{a^2+x^2}\,dx = \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right| + \frac{x}{2}\sqrt{a^2+x^2} + c$ (Ans)

Ex 50: Integrate: $\int\sqrt{x^2-a^2}\,dx$.

Soln: Let, $I = \int\sqrt{x^2-a^2}\,dx$

$= \sqrt{x^2-a^2}\int dx - \int\left(\int dx\right)\cdot\left(\frac{d}{dx}\sqrt{x^2-a^2}\right)dx$

$= \sqrt{x^2-a^2}\cdot x - \int x\cdot\frac{1}{2\sqrt{x^2-a^2}}\cdot 2x\,dx$

$= x\cdot\sqrt{x^2-a^2} - \int\frac{x^2}{\sqrt{x^2-a^2}}\,dx = x\sqrt{x^2-a^2} - \int\frac{(x^2-a^2)+a^2}{\sqrt{x^2-a^2}}\,dx$

$= x\sqrt{x^2-a^2} - \int\sqrt{x^2-a^2}\,dx - a^2\int\frac{dx}{\sqrt{x^2-a^2}}$

$= x\sqrt{x^2-a^2} - I - a^2\ln\left|x+\sqrt{x^2-a^2}\right| + c_1$

$\Rightarrow 2I = x\sqrt{x^2-a^2} - a^2\ln|x+\sqrt{x^2-a^2}|+c_1$

$\Rightarrow I = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}|+\frac{c_1}{2}$

$\Rightarrow \int\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}|+C$ (ANS)

51: Integrate: $\int e^{ax}\sin bx\,dx$.

Soln: Let $I = \int e^{ax}\sin bx\,dx = \sin bx\int e^{ax}dx - \int\left(\int e^{ax}dx \times \frac{d}{dx}\sin bx\right)dx$

(Integrating by parts)

$= \frac{e^{ax}}{a}\cdot\sin bx - \int\frac{e^{ax}}{a}\cdot b\cos bx\,dx = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int e^{ax}\cos bx\,dx$.

$= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\left[\cos bx\int e^{ax}dx - \int\left(\int e^{ax}dx\right)\times\left(\frac{d}{dx}\cos bx\right)dx\right]$

$= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\left[\frac{1}{a}e^{ax}\cos bx - \int\frac{1}{a}e^{ax}\cdot(-b\sin bx)dx\right]$

$= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx - \frac{b^2}{a^2}\int e^{ax}\sin bx\,dx$

$= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx - \frac{b^2}{a^2}I + c_1$

$= \frac{e^{ax}}{a^2}\left[a\sin bx - b\cos bx\right] - \frac{b^2}{a^2}I + c_1$

$\Rightarrow I + \frac{b^2}{a^2}I = \frac{e^{ax}}{a^2}\left[a\sin bx - b\cos bx\right] + c_1$

$\Rightarrow \frac{a^2+b^2}{a^2}\cdot I = \frac{e^{ax}}{a^2}\left[a\sin bx - b\cos bx\right] + c_1$

$\Rightarrow I = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right] + \frac{c_1a^2}{a^2+b^2}$

$\Rightarrow \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right] + C$ (ANS)

Where $C = \frac{c_1a^2}{a^2+b^2}$

Ex:52 Integrate: $\int e^{2x}\cos 3x\,dx$.

Soln: Let $I = \int e^{2x}\cos 3x\,dx$

$= \cos 3x\int e^{2x}dx - \int\left(\int e^{2x}dx\right)\times\left(\frac{d}{dx}\cos 3x\right)dx$

(Integrating by parts)

$= \frac{1}{2}e^{2x}\cos 3x - \int\frac{1}{2}e^{2x}\times(-3\sin 3x)\,dx$

$= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\int e^{2x}\sin 3x\,dx$.

$= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\left[\sin 3x\int e^{2x}dx - \int\left(\int e^{2x}dx\right)\times\left(\frac{d}{dx}\sin 3x\right)dx\right]$

$= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\left[\frac{1}{2}e^{2x}\sin 3x - \int\frac{1}{2}e^{2x}\times 3\cos 3x\,dx\right]$

$= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x - \frac{9}{4}\int e^{2x}\cos 3x\,dx$.

$= \frac{e^{2x}(2\cos 3x + 3\sin 3x)}{4} - \frac{9}{4}I + c_1$

$$\Rightarrow I+\frac{9}{4}I=\frac{e^{2x}(2\cos 3x+3\sin 3x)}{4}+c_1$$

$$\Rightarrow \frac{13I}{4}=\frac{e^{2x}(2\cos 3x+3\sin 3x)}{4}+c_1$$

$$\Rightarrow I=\frac{e^{2x}}{13}[2\cos 3x+3\sin 3x]+\frac{4c_1}{13}$$

$$\Rightarrow \int e^{2x}\cos 3x\,dx=\frac{e^{2x}}{13}[2\cos 3x+3\sin 3x]+c$$

Taking $c=\frac{4c_1}{13}$ (Ans).

EXERCISE:

Integrate following:

(1) $\int \frac{dx}{x^2+16}$ (5 marks)

(2) $\int \frac{dx}{x^2-a^2}$

(3) $\int \frac{dx}{16-x^2}$ (5 marks)

(4) $\int \frac{dx}{\sqrt{x^2+a^2}}$

(5) $\int \frac{dx}{\sqrt{x^2-9}}$ (5 marks)

(6) $\int \frac{dx}{\sqrt{4-x^2}}$ (5 marks)

(7) $\int \frac{dx}{x\sqrt{x^2-9}}$ (5 marks)

(8) $\int \sqrt{36-x^2}\,dx$ (5 marks)

(9) $\int \sqrt{64+x^2}\,dx$ (5 marks)

(10) $\int \sqrt{x^2-9}\,dx$ (5 marks)

(11) $\int e^{2x}\sin 5x\,dx$ (5 marks)

(12) $\int e^{ax}\cos bx\,dx$. (5 marks)

J: INTEGRATION USING PARTIAL FRACTION:

For integration of a rational fractions of the form $\frac{P(x)}{Q(x)}$, the process of partial fraction can be used which has discussed in 1st. Semester.

Ex-53: Integrate $\int \frac{(3x+1)}{(x+1)(x-2)} dx$.

Soln: Let $\frac{3x+1}{(x+1)(x-2)} = \frac{A}{(x+1)} + \frac{B}{x-2}$ ——— ①

$\Rightarrow \quad 3x+1 = A(x-2) + B(x+1)$ ——— ②

put $x = -1$ in Eqn (2)

$\Rightarrow \quad 3(-1)+1 = A(-1-2) \Rightarrow -2 = -3A \Rightarrow \boxed{A = \frac{2}{3}}$

put $x = 2$ in Eqn (2) $\Rightarrow 3(2)+1 = B(2+1)$

$\Rightarrow \quad 7 = 3B \Rightarrow \boxed{B = \frac{7}{3}}$

From Eqn (1) $\frac{3x+1}{(x+1)(x-2)} = \frac{2/3}{x+1} + \frac{7/3}{x-2}$ ——— (3)

Now $\int \frac{3x+1}{(x+1)(x-2)} dx = \int \left(\frac{2/3}{x+1} + \frac{7/3}{x-2}\right) dx$ using Eqn (3)

$= \frac{2}{3} \int \frac{dx}{x+1} + 7/3 \int \frac{dx}{x-2} = \frac{2}{3} \ln|x+1| + \frac{7}{3} \ln|x-2| + c$ (Ans.)

EXERCISE:

INTEGRATE FOLLOWING:

① $\int \frac{3x-4}{(x-1)^2(x+1)} dx$

(2) $\int \frac{dx}{x^2-a^2}$

(3) $\int \frac{dx}{a^2-x^2}$

(4) $\int \frac{dx}{(x-1)(x-2)}$

(5) $\int \frac{dx}{(x^2+4)(x-3)}$

6) $\int \frac{dx}{(x^2+1)(x^2+9)}$

DEFINITE INTEGRAL:

→ Integration can be considered as process of summation and in this case the integral is called definite integral.

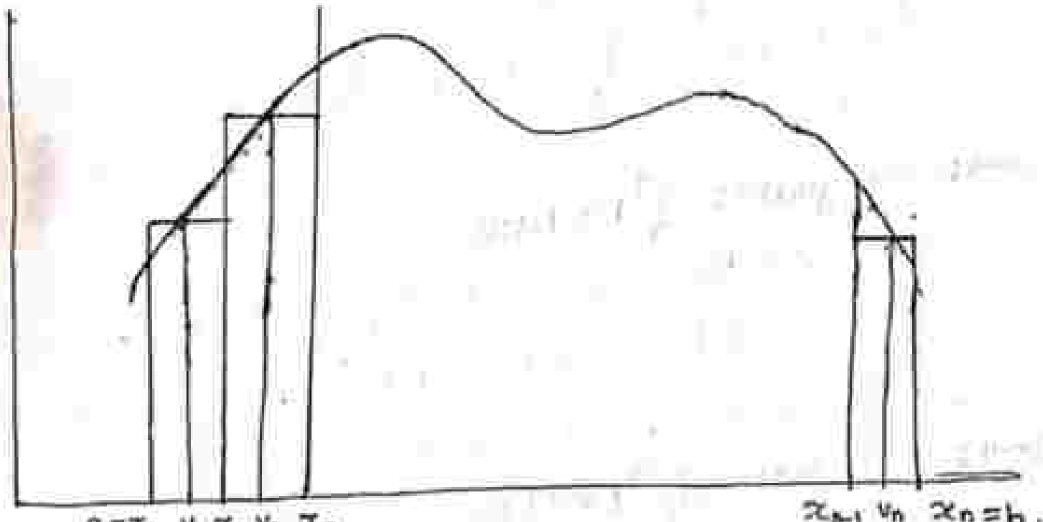
→ Let $f(x)$ be a continuous function defined on $[a,b]$ at $(n+1)$ distinct points $x_0, x_1, x_2 ----- x_n$ s.t $a \leq x_0 < x_1 < ...... \leq x_n \leq b$ where a and b are finite. Divide $[a,b]$ into 'n' sub intervals of interval lengths $h_1, h_2 ----- h_n$ where $h_i = x_i - x_{i-1}, 1 \leq i \leq n$. Let v_r be any point in $[x_{r-1}, x_r]$ i.e $v_1 \in [x_0, x_1]$, $v_2 \in [x_1, x_2]$ ...... $v_n \in [x_{n-1}, x_n]$. Then sum of areas of rectangles of lengths $f(v_r)$ and breadth h_r, when $n \to \infty$ shown in fig. is called definite integral of $f(x)$ from a to b denoted by $\int_a^b f(x)dx$.

a = lower limit of integration, b = upper limit of integration.

Mathematically,

$$\int_a^b f(x)dx = \lim_{n\to\infty} [h_1 f(v_1) + h_2 f(v_2) + ... h_n f(v_n)] =$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} h_i f(v_i)$$



K1: FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS:

If $f(x)$ is a continuous function on $[a,b]$ s.t $\int f(x)dx = F(x)+c$ then $\int_a^b f(x)dx = F(b) - F(a)$.

K2: Properties:

(i) Definite integral does not depend on constant of integration.

(ii) When we use substitution to find $\int_a^b f(x)dx$, we have to find out the new limits of integration of new variable corresponding to limit of integration of old variable.

(iii) $\int_a^b f(x)dx = \int_a^b f(t)dt$ (iv) $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

(v) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \quad a < c < b$

(vi) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ · (vii) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

(viii) $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ if $f(-x) = f(x)\ \forall x$

$= 0$ if $f(-x) = -f(x), \forall x$.

(ix) $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$ if $f(2a-x) = f(x)$

$= 0$ if $f(2a-x) = -f(x)$.

Ex: 54 Integrate : $\int_0^1 \frac{dx}{1+x^2}$.

Soln: $\int_0^1 \frac{dx}{1+x^2} = [\tan^{-1}x]_0^1 = \tan^{-1}1 - \tan^{-1}0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$ (Ans)

Ex: 55. Integrate: $\int_2^3 2xe^{x^2}dx$.

Soln: $\int_2^3 2x\, e^{x^2}dx = \int_4^9 e^t dt = [e^t]_4^9$

$= e^9 - e^4$ (Ans)

put $x^2 = t$
$\Rightarrow 2x dx = dt$
At $x = 2, t = 4$
$x = 3, t = 9$

Ex-56: Integrate: $\int_1^4 [x]dx$.

Soln: $\int_1^4 [x]dx = \int_1^2 [x]dx + \int_2^3 [x]dx + \int_3^4 [x]dx$.

$= \int_1^2 1.dx + \int_2^3 2.dx + \int_3^4 3.dx = [x]_1^2 + [2x]_2^3 + [3x]_3^4$

$= [2-1] + [6-4] + [12-9] = 1+2+3 = 6$ (Ans)

Ex-57: Integrate: $\int_0^{3/2} [2x]dx$

Soln: $\int_0^{3/2} [2x]dx = \int_0^3 [t].\frac{1}{2}dt$

$= \frac{1}{2}\int_0^3 [t]dt = \frac{1}{2}\left\{\int_0^1 [t]dt + \int_1^2 [t]dt + \int_2^3 [t]dt\right\}$

$= \frac{1}{2}\left\{\int_0^1 0.dt + \int_1^2 1.dt + \int_2^3 2dt\right\}$

$= \frac{1}{2}\left\{[t]_1^2 + [2t]_2^3\right\} = \frac{1}{2}\left\{(2-1) + (6-4)\right\}$

$= \frac{1}{2} \times 3 = \frac{3}{2}$ (Ans)

put $2x = t$
$\Rightarrow 2dx = dt$
$\Rightarrow dx = \frac{1}{2}dt$
At $x = 0, t = 0$
$x = \frac{3}{2}, t = 3$

Ex:58 Integrate: $\int_0^2 [x^2]dx$

Solⁿ: $\int_0^2 [x^2]dx = \int_0^4 [t].\frac{dt}{2\sqrt{t}} = \frac{1}{2}\int_0^4 \frac{[t]}{\sqrt{t}}dt$

(putting $x^2 = t \Rightarrow 2x\,dx = dt \Rightarrow dx = \frac{dt}{2x} = \frac{dt}{2\sqrt{t}}$
At $x=0, t=0$ and $x=2, t=4$)

$$= \frac{1}{2}\left[\int_0^1 \frac{[t]}{\sqrt{t}}dt + \int_1^2 \frac{[t]}{\sqrt{t}}dt + \int_2^3 \frac{[t]}{\sqrt{t}}dt + \int_3^4 \frac{[t]}{\sqrt{t}}dt\right]$$

$$= \frac{1}{2}\left[\int_0^1 \frac{0}{\sqrt{t}}dt + \int_1^2 \frac{1}{\sqrt{t}}dt + \int_2^3 \frac{2}{\sqrt{t}}dt + \int_3^4 \frac{3}{\sqrt{t}}dt\right]$$

$$= \frac{1}{2}\left\{\int_1^2 t^{-\frac{1}{2}}dt + 2\int_2^3 t^{-\frac{1}{2}}dt + 3\int_3^4 t^{-\frac{1}{2}}dt\right\}$$

$$= \frac{1}{2}\left\{[2\sqrt{t}]_1^2 + 2[2\sqrt{t}]_2^3 + 3[2\sqrt{t}]_3^4\right\}$$

$$\left(\because \int t^{-\frac{1}{2}}dt = \frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \frac{t^{1/2}}{1/2} = 2\sqrt{t}\right)$$

$$= \frac{1}{2}\left\{2\sqrt{2} - 2 + 4\sqrt{3} - 4\sqrt{2} + 12 - 6\sqrt{3}\right\}$$

$$= \frac{1}{2}\left\{-2\sqrt{2} - 2\sqrt{3} + 10\right\} = 5 - \sqrt{2} - \sqrt{3} \text{ (Ans)}$$

Ex-59: Integrate: $\int_{0.5}^2 [x]dx$.

Solⁿ: $\int_{0.5}^2 [x]dx = \int_{0.5}^1 [x]dx + \int_1^2 [x]dx = \int_{0.5}^1 0\cdot dx + \int_1^2 1\cdot dx = [x]_1^2 = 2-1 = 1$

Ex-60: Integrate: $\int_3^5 \frac{|x|}{[x]}dx$.

Solⁿ: $\int_3^5 \frac{|x|}{[x]}dx = \int_3^4 \frac{|x|}{[x]}dx + \int_4^5 \frac{|x|}{[x]}dx = \int_3^4 \frac{x}{3}dx + \int_4^5 \frac{x}{4}dx$

$= \frac{1}{3}\int_3^4 x\,dx + \frac{1}{4}\int_4^5 x\,dx = \frac{1}{3}\left[\frac{x^2}{2}\right]_3^4 + \frac{1}{4}\left[\frac{x^2}{2}\right]_4^5 = \frac{1}{6}(16-9) + \frac{1}{8}(25-16)$

$= \frac{7}{6} + \frac{9}{8} = \frac{56+54}{48} = \frac{110}{48} = \frac{55}{24}$ (Ans).

Ex-61 Integrate: $\int_{-2}^1 (|x|+x)dx$.

Solⁿ: $\int_{-2}^1 (|x|+x)dx = \int_{-2}^0 (|x|+x)dx + \int_0^1 (|x|+x)dx$.

$= \int_{-2}^0 (-x+x)dx + \int_0^1 (x+x)dx = \int_0^1 2x\,dx = 2\cdot\left[\frac{x^2}{2}\right]_0^1 = 2\left[\frac{1}{2} - 0\right]$

$= 2\times\frac{1}{2} = 1$ (Ans).

Ex-62 Integrate: $\int_0^1 \frac{dx}{e^x + e^{-x}}$

Solⁿ: $\int_0^1 \frac{dx}{e^x + e^{-x}} = \int_0^1 \frac{1}{e^x + \frac{1}{e^x}}dx = \int_0^1 \frac{e^x\,dx}{e^{2x}+1} = \int_0^1 \frac{e^x\,dx}{(e^x)^2+1}$

$$= \int_1^e \frac{dt}{t^2+1} = [\tan^{-1}t]_1^e = \tan^{-1}e - \tan^{-1}1$$ $\left[\begin{array}{l}\text{put } e^x = t \Rightarrow e^x dx = dt \\ \text{At } x=0, t=1 \text{ \& } x=1, t=e\end{array}\right.$

$$= \tan^{-1}e - \frac{\pi}{4} \quad \text{(Ans)}$$

Ex:63 $\int_0^1 x\log(1+x)dx$.

Soln: $\int_0^1 x\log(1+x)dx = \left[\log(1+x).\int x dx - \int\left(\int x dx\right).\frac{d}{dx}\log(1+x).dx\right]_0^1$

(Integrating by parts.)

$$= \left[\log(1+x)\cdot\frac{x^2}{2} - \int\frac{x^2}{2}\times\frac{1}{1+x}\times 1\cdot dx\right]_0^1$$

$$= \left[\frac{x^2\log(1+x)}{2} - \frac{1}{2}\int\frac{x^2}{1+x}dx\right]_0^1$$

$$= \left[\frac{x^2.\log(1+x)}{2} - \frac{1}{2}\int\frac{(x^2-1)+1}{(1+x)}dx\right]_0^1$$

$$= \left[\frac{x^2\log(1+x)}{2} - \frac{1}{2}\int\frac{(x+1)(x-1)+1}{(1+x)}dx\right]_0^1$$

$$= \left[\frac{x^2\log(1+x)}{2} - \frac{1}{2}\int\left(x-1+\frac{1}{1+x}\right)dx\right]_0^1$$

$$= \left[\frac{x^2\log(1+x)}{2} - \frac{1}{2}\int x dx + \frac{1}{2}\int dx - \frac{1}{2}\int\frac{dx}{1+x}\right]_0^1$$

$$= \left[\frac{x^2\log(1+x)}{2} - \frac{1}{2}\times\frac{x^2}{2} + \frac{1}{2}\times x - \frac{1}{2}\ln|(1+x)|\right]_0^1$$

$$= \left[\frac{x^2\log(1+x)}{2} - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2}\ln|(1+x)|\right]_0^1$$

$$= \left[\left(\frac{\log 2}{2} - \frac{1}{4} + \frac{1}{2} - \frac{\log 2}{2}\right) - (0-0+0-0)\right]$$

$$= \frac{1}{2} - \frac{1}{4} = \frac{2-1}{4} = \frac{1}{4} \quad \text{(Ans)}.$$

Ex64: Integrate: $\int_0^{\pi/2}\frac{\sin x}{\sin x+\cos x}dx$.

Soln: Let $I = \int_0^{\pi/2}\frac{\sin x}{\sin x+\cos x}dx$ ---------(i)

$$= \int_0^{\pi/2}\frac{\sin(\frac{\pi}{2}-x)}{\sin(\frac{\pi}{2}-x)+\cos(\pi/2-x)}dx \quad \because \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$= \int_0^{\pi/2}\frac{\cos x}{\cos x+\sin x}dx \quad \text{-------- (ii)}$$

Now

$$2I = I+I = \int_0^{\pi/2}\frac{\sin x}{\sin x+\cos x}dx + \int_0^{\pi/2}\frac{\cos x}{\cos x+\sin x}dx$$

$$= \int_0^{\pi/2}\left(\frac{\sin x}{\sin x+\cos x} + \frac{\cos x}{\cos x+\sin x}\right)dx$$

$$= \int_0^{\pi/2}\left(\frac{\sin x+\cos x}{\sin x+\cos x}\right)dx = \int_0^{\pi/2}dx = [x]_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4} \Rightarrow \int_0^{\pi/2}\frac{\sin x}{\sin x+\cos x}dx = \frac{\pi}{4} \quad \text{(Ans)}.$$

Ex-65: $\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx.$

Soln: $I=\int_0^1 \frac{\ln(1+x^2)}{1+x^2}dx = \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan\theta)}{1+\tan^2\theta}\cdot\sec^2\theta\, d\theta$ put $x=\tan\theta$ $\Rightarrow dx = \sec^2\theta\, d\theta$

At $x=0, \theta=0$
At $x=1, \theta=\frac{\pi}{4}$

$= \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan\theta)}{\sec^2\theta}\cdot\sec^2\theta\, d\theta = \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,d\theta \;\ldots\ldots (t)$

$= \int_0^{\frac{\pi}{4}} \ln[1+\tan(\frac{\pi}{4}-\theta)]d\theta = \int_0^{\frac{\pi}{4}} \ln\left[1+\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}.\tan\theta}\right]d\theta$

$= \int_0^{\frac{\pi}{4}} \ln\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right]d\theta = \int_0^{\frac{\pi}{4}} \ln\left[\frac{1+\tan\theta+1-\tan\theta}{1+\tan\theta}\right]d\theta$

$= \int_0^{\pi/4} \ln\left[\frac{2}{1+\tan\theta}\right]d\theta = \int_0^{\frac{\pi}{4}} [\ln 2 - \ln(1+\tan\theta)]d\theta$

$= \int_0^{\frac{\pi}{4}} \ln 2\cdot d\theta - \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)d\theta = \ln 2\int_0^{\pi/4} d\theta - I$ using (t)

$\Rightarrow 2I = \ln 2\,[\theta]_0^{\frac{\pi}{4}} = \ln 2.[\frac{\pi}{4}-0] = \ln 2\cdot\frac{\pi}{4}$

$\Rightarrow I = \frac{\pi}{8}\ln 2 \Rightarrow \int_0^1 \frac{\ln(1+x)}{1+x^2}dx = \frac{\pi}{8}\ln 2$ (Ans)

Ex-66: Evaluate: $\int_0^{\frac{\pi}{2}} \log(\sin x)dx.$

Soln: Let $I = \int_0^{\frac{\pi}{2}} \log(\sin x)dx \;\text{---------} (i)$

$= \int_0^{\frac{\pi}{2}} \log[\sin(\frac{\pi}{2}-x)]dx = \int_0^{\frac{\pi}{2}} \log(\cos x)dx \;\text{------} (ii)$

Now $2I = I+I = \int_0^{\pi/2} \log(\sin x)dx + \int_0^{\frac{\pi}{2}} \log(\cos x)dx$

$= \int_0^{\frac{\pi}{2}} [\log(\sin x)+\log(\cos x)]dx. = \int_0^{\frac{\pi}{2}} \log(\sin x.\cos x)\,dx$

$= \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin 2x}{2}\right)dx = \int_0^{\frac{\pi}{2}} \log(\sin 2x)dx - \int_0^{\pi/2} \log 2\cdot dx.$

$= \int_0^{\pi} \log(\sin t).\frac{1}{2}dt - \log 2\cdot\int_0^{\frac{\pi}{2}} dx$

(putting $2x=t \Rightarrow dx=\frac{1}{2}dt$ and $x=0\Rightarrow t=0, x=\frac{\pi}{2}\Rightarrow t=\pi$)

$= \frac{1}{2}\int_0^{\pi} \log(\sin t)dt - \log 2\,[x]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\int_0^{\pi} \log(\sin x)dx - \log 2\times[\frac{\pi}{2}-0] = \frac{1}{2}\int_0^{2\times\frac{\pi}{2}} \log(\sin x)dx - \frac{\pi}{2}\log 2.$

$= \frac{1}{2}\times 2\times\int_0^{\frac{\pi}{2}} \log(\sin x)dx - \frac{\pi}{2}\log 2$ $\because \log[\sin(2\times\frac{\pi}{2}-x)] = \log(\sin x)$

$= \int_0^{\pi/2} \log(\sin x)dx - \frac{\pi}{2}\log 2 = I - \frac{\pi}{2}\log 2.$

$\Rightarrow 2I - I = -\frac{\pi}{2}\log 2 \Rightarrow I = -\frac{\pi}{2}\log 2 \Rightarrow \int_0^{\pi/2} \log(\sin x)dx = -\frac{\pi}{2}\log 2$

(Ans)

EXERCISE

Integrate:

(1) $\int_1^2 x^3 dx$ (2) $\int_0^2 |x-1| dx$ (3) $\int_{-2}^{2} (|x| + [x]) dx$ (4) $\int_{3}^{5} |x| dx$

(5) $\int_{-4}^{-3} |x| dx$ (6) $\int_0^{\frac{\pi}{2}} \sin x \, dx$ (7) $\int_0^{\frac{\pi}{2}} x \cos x \, dx$ (8) $\int_0^{\pi/4} \sin^5 x \cos x$

(9) $\int_{-3}^{4} |x| dx$ (10) $\int_{-2}^{2} (x + |x|) dx$ (11) $\int_0^1 [x] dx$ (12) $\int_0^1 [x+1] dx$.

(13) $\int_1^3 [x] dx$ (14) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ (15) $\int_{-1}^{2} [x] dx$ (16) $\int_1^{\sqrt{2}} \frac{dx}{x\sqrt{x^2-1}}$

(17) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x} dx$ (18) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan x}$ (19) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cot x}$

(20) $\int_0^{\pi/2} \frac{\tan x}{\tan x + \cot x} dx$ (21) $\int_0^{\frac{\pi}{2}} \frac{\cot x}{\tan x + \cot x}$ (22) $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

(23) $\int_0^{\pi} \frac{x \, dx}{1+\sin x}$ (24) $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$ (25) $\int_0^{\pi/2} \frac{dx}{1+\sin x}$

(26) $\int_0^{\pi/2} \log(\tan x) dx$ (27) $\int_0^{\pi/2} \log(1+\cot\theta) d\theta$ (28) $\int_0^{\pi/2} \log(\cos x)$

L: WALLI'S INTEGRAL FORMULA:

This formula is attributed to English Mathematician John Wallis Which is stated below:

If $m, n \in \mathbb{N}$ then

(i) $\int_0^{\pi/2} \sin^m x \cdot \cos^n x \, dx = \int_0^{\pi/2} \sin^n x \cos^m x \, dx$

$$= \begin{cases} \dfrac{(m-1)(m-3)\cdots(1 \text{ or } 2)(n-1)(n-3)\cdots(1 \text{ or } 2)}{(m+n)(m+n-2)(m+n-4)\cdots(1 \text{ or } 2)} \times \dfrac{\pi}{2}, & \text{if } m \text{ and } n \text{ both even} \\ \dfrac{(m-1)(m-3)\cdots(1 \text{ or } 2)(n-1)(n-3)\cdots(1 \text{ or } 2)}{(m+n)(m+n-2)(m+n-4)\cdots(1 \text{ or } 2)}, & \text{otherwise.} \end{cases}$$

(ii) $\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_0^{\frac{\pi}{2}} \cos^n x \, dx = \begin{cases} \dfrac{(n-1)(n-3)(n-5)\cdots 3 \cdot 1}{n(n-2)(n-4)\cdots 4 \cdot 2} \dfrac{\pi}{2}, & n \text{ even} \\ \dfrac{(n-1)(n-3)(n-5)\cdots 4 \cdot 2}{n(n-2)(n-4)\cdots 5 \cdot 3}, & n \text{ is odd} \end{cases}$

Ex-69: Evaluate $\int_0^{\frac{\pi}{2}} \sin^6 x \cdot \cos^2 x \, dx$

Soln: $\int_0^{\frac{\pi}{2}} \sin^6 x \cdot \cos^2 x \, dx$

$= \dfrac{(6-1)(6-3)(6-5)(2-1)}{(6+2)(6+2-2)(6+2-4)(6+2-6)} \cdot \dfrac{\pi}{2}$ ($\because m=6, n=2$ both even and using Wallis Formula)

$$= \frac{5\times3\times1\times1}{8\times6\times4\times2}\times\frac{\pi}{2} = \frac{15\pi}{768} \text{ (Ans)}$$

Ex-68: Evaluate $\int_{-\pi/2}^{\pi/2} \sin^4 x \cos^6 x \, dx$

Soln: $\int_{-\pi/2}^{\pi/2} \sin^4 x \cdot \cos^6 x \, dx = 2\int_0^{\pi/2} \sin^4 x \cdot \cos^6 x \, dx$ $\because \sin^4 x \cdot \cos^6 x$ is an even function.

$$= 2\times \frac{(4-1)(4-3)(6-1)(6-3)(6-5)}{(4+6)(4+6-2)(4+6-4)(4+6-6)(4+6-8)}\frac{\pi}{2} \quad \because m=4, n=6 \text{ even.}$$

$$= \frac{2\times3\times1\times5\times3\times1}{10\times8\times6\times4\times2}\times\frac{\pi}{2} = \frac{3\pi}{256} \text{ (Ans).}$$

Ex-69: $\int_0^{\pi/2} \cos^7 x \, dx$

Soln: $\int_0^{\pi/2} \cos^7 x \, dx = \frac{(7-1)(7-3)(7-5)}{7(7-2)(7-4)} = \frac{6\times4\times2}{7\times5\times3}$ $(n=7 \text{ odd})$

$$= \frac{16}{35} \text{ (Ans)}$$

Ex-70: Evaluate $\int_0^{\pi/2} \sin^7 x \cdot \cos^8 x \, dx$.

Soln: $\int_0^{\pi/2} \sin^7 x \cdot \cos^8 x \, dx = \frac{(7-1)(7-3)(7-5)\cdot(8-1)(8-3)(8-5)(8-7)}{(7+8)(7+8-2)(7+8-4)(7+8-6)(7+8-8)(7+8-10)\times(7+8-12)(7+8-14)}$

$(\because m=7, n=8, m \,\&\, n \text{ are not both even})$

$$= \frac{6\times4\times2\times7\times5\times3\times1}{15\times13\times11\times9\times7\times5\times3\times1} = \frac{16}{6435} \text{ (Ans)}$$

EXERCISE:

Evaluate:

(1) $\int_0^{\pi/2} \sin^3 x \cdot \cos^4 x \, dx$ (2) $\int_0^{\pi/2} \sin^8 x \cdot \cos^5 x \, dx$ (3) $\int_0^{\pi/2} \sin^8 x \, dx$

(4) $\int_0^{\pi/2} \cos^6 x \, dx$ (5) $\int_0^{\pi/2} \sin^5 x \, dx$ (6) $\int_0^{\pi/2} \cos^9 x \, dx$ (7) $\int_{-\pi/2}^{\pi/2} \sin^6 x \cos^3 x \, dx$

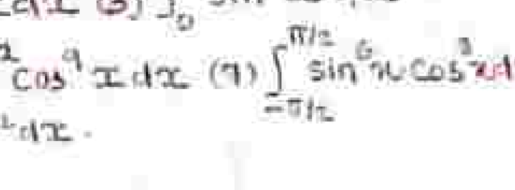
(8) $\int_0^a x^2 (a^2-x^2)^{3/2} dx$.

M: AREA BOUNDED BY PLANE CURVES:

(i) The process of finding Area of a plane figure is called Quadrature.

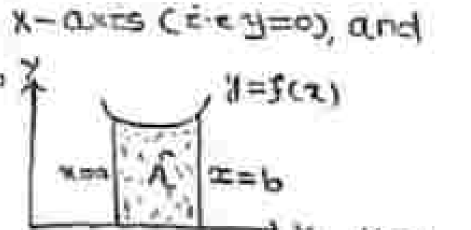
(ii) Area bounded by the curve $y=f(x)$, x-axis (i.e $y=0$), and ordinates $x=a, x=b$ is

$$\boxed{A = \int_a^b y\,dx = \int_a^b f(x)\,dx, \text{ Take +ve value}}$$



(iii) Area bounded by the curve $x=f(y)$, Y-axis (i.e $x=0$) and abscissas $y=a, y=b$ is

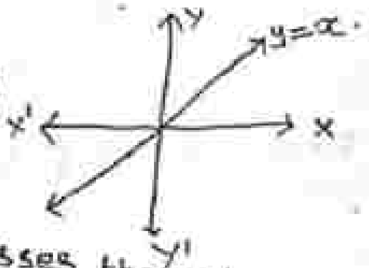
$$\boxed{A = \int_a^b x\,dy = \int_a^b f(y)\,dy, \text{ +ve value.}}$$



NOTE: → Equation of circle with centre at origin and radius 'a' is $\boxed{x^2+y^2=a^2}$

→ To find area of a plane region we should rember following points.

* Area of symmetrical regions about a line are equal.
* Rough sketch: To find area of a region we have to sketch the curve Roughly by finding some points (x,y) which are satisfying the equation of curve.
* Symmetry of the curve should be checked by following rules:
* If Equation of curve contains only even powers of x, then the curve is symmetrical about y-axis.
  Ex: $x^2=y$ is symmetrical about y-axis.
* If equation of curve contains only even powers of y, then the curve is symmetrical about x-axis.
  Ex: $y^2=4x$ is symmetrical about x-axis.
* If powers of both x and y are even, then the curve is symmetrical about both axes.
  Ex: $x^2+y^2=4$ is symmetrical about both axes.
* If Equation of curve remains unchanged by interchanging x and y, then the curve is symmetrical about the line $y=x$.
  Ex: $x^3+y^3=3axy$ is symmetrical about $y=x$ line.



* If $x=0 \Rightarrow y=0$ then the curve passes through origin.
  If $x=0 \Rightarrow y=a$ then the curve cuts Y-axis at $(0,a)$
  If $y=0 \Rightarrow x=a$ then the curve cuts x-axis at $(a,0)$.

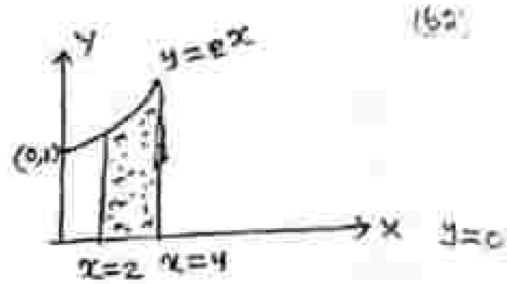
Ex: Find the area bounded by $y=e^x$, $y=0$, $x=4$, $x=2$.

Soln: $y=e^x$ —— ①

If $x=0 \Rightarrow y=1$

Required Area $= \int_2^4 y\,dx = \int_2^4 e^x dx$

$= [e^x]_2^4 = e^4 - e^2$ sq unit (Ans)



Ex-72 Find the area bounded by the curve $xy=c^2$, x-axis, $x=2$, $x=3$.

Soln: $xy=c^2 \Rightarrow y=\frac{c^2}{x}$ —— ①

Required Area $= \int_2^3 y.dx = \int_2^3 \frac{c^2}{x}dx = c^2\int_2^3 \frac{dx}{x}$

$= c^2[\ln|x|]_2^3 = c^2(\ln|3|-\ln|2|) = c^2(\ln 3-\ln 2) = c^2\ln\left(\frac{3}{2}\right)$ sq. unit

Ex-73 Find Area of the circle $x^2+y^2=16$

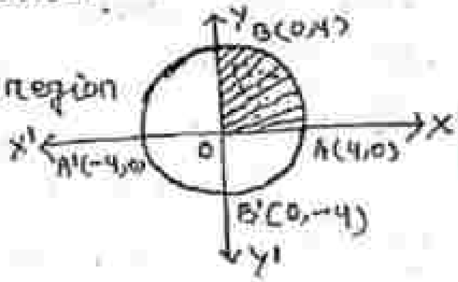
Soln: $x^2+y^2=16 \Rightarrow y=\sqrt{16-x^2}$ —— ①

If $x=0 \Rightarrow y=\pm 4$

and if $y=0 \Rightarrow x=\pm 4$

Hence the circle cuts x-axis at $A(4,0)$, $A'(-4,0)$ and y-axis at $B(0,4)$, $B'(0,-4)$. It is symmetrical about both axes.

Required Area $= 4\times$ Area of shaded region



$= 4\times\int_0^4 y\,dx = 4\times\int_0^4 \sqrt{16-x^2}\,dx$

$= 4\times\int_0^4 \sqrt{(4)^2-x^2}\,dx$

$= 4\times\left[\frac{4^2}{2}\sin^{-1}\frac{x}{4} + \frac{x}{2}\sqrt{4^2-x^2}\right]_0^4$ $\quad\because \int\sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}$

$= 4\times\left[8\sin^{-1}\frac{x}{4} + \frac{x}{2}\sqrt{16-x^2}\right]_0^4$

$= 4\times[(8\sin^{-1}1 + 2\times 0) - 0] = 32\sin^{-1}1 = 32\times\frac{\pi}{2} = 16\pi$ sq. units (Ans)

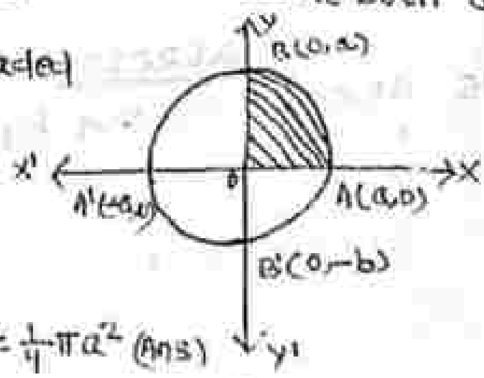
Ex-74 Find Area of a quadrant of the circle $x^2+y^2=a^2$.

Soln: $x^2+y^2=a^2 \Rightarrow y=\sqrt{a^2-x^2}$ —— ①

If $x=0 \Rightarrow y=\pm a$ and If $y=0 \Rightarrow x=\pm a$

Hence the circle cuts x-axis at $A(a,0)$, $A'(-a,0)$ and y-axis at $B(0,a)$, $B'(0,-a)$, It is symmetrical about both axes.

Required Area = Area of the shaded region



$= \int_0^a y\,dx = \int_0^a \sqrt{a^2-x^2}\,dx$

$= \left[\frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2}\right]_0^a$

$= \left[\frac{a^2}{2}\sin^{-1}1 - 0\right] = \frac{a^2}{2}\times\frac{\pi}{2} = \frac{1}{4}\pi a^2$ (Ans)

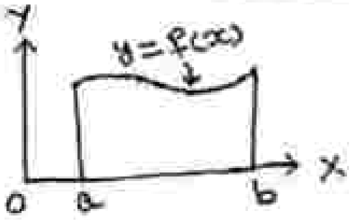
Exercises:

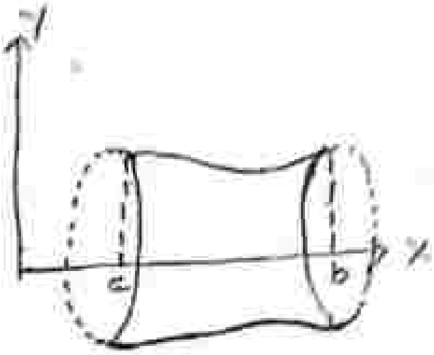
1) Find the area bounded by the curve $x^2+y^2=4$
2) Find the area of the circle $x^2+y^2=9$
3) Find the area enclosed by $y=4-x^2$, $y=0$, $x=0$, $x=2$.
4) Evaluate the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
5) Find the area enclosed by one quadrant of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$

N: VOLUME OF SOLID BY REVOLUTION OF AN AREA ABOUT AXES:

(i) The volume of a solid formed by revolution of an area about x-axis bounded by the curve $y=f(x)$, the ordinates $x=a$, $x=b$ is

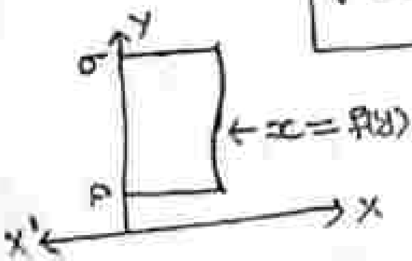
$$\boxed{V=\int_a^b \pi y^2\,dx}$$

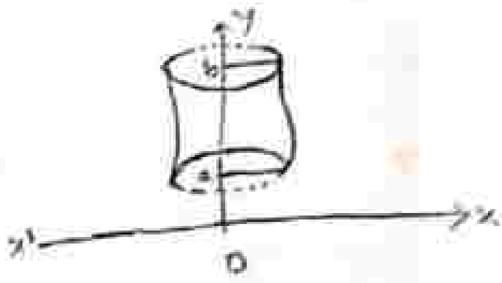




(ii) The volume of a solid formed by revolution of an area about y-axis bounded by the curve $x=f(y)$, the abscissas $y=a$, $y=b$ is

$$\boxed{V=\int_a^b \pi x^2\,dy}$$



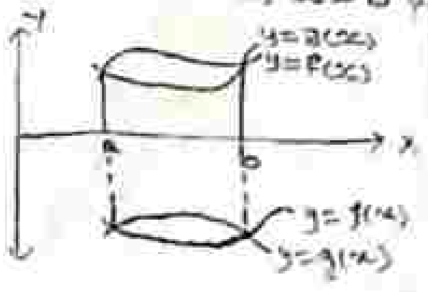


(iii) The volume of the solid formed by rotating the area between the curves $y=f(x)$ and $y=g(x)$ and ordinates $x=a$, $x=b$ about x-axis is

$$\boxed{V=\int_a^b \pi\left[f(x)^2-g(x)^2\right]dx}$$

Similarly about y-axis.

In all above cases we take volume as +ve.

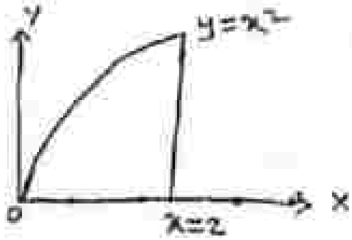


Ex-75: Let $y = f(x) = x^2$ be the curve given on the interval $[0, 2]$. Find the volume of the solid generated by revolving the region between the curve in the given interval around x-axis.

Soln: $y = f(x) = x^2$ —— (1)

ordinates are $x = 0, x = 2$

Required volume



$$V = \int_0^2 \pi y^2 dx = \int_0^2 \pi (x^2)^2 dx$$

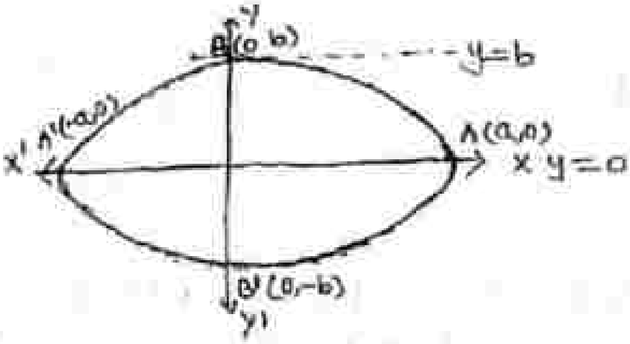
$$= \pi \int_0^2 x^4 dx$$

$$= \pi \left[\frac{x^5}{5} \right]_0^2 = \frac{\pi}{5}[32 - 0] = \frac{32\pi}{5} \text{ cubic unit (Ans)}$$

Ex-76: Evaluate the volume of the solid generated by the revolution of an Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about its minor axis.

Soln: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $b < a$ and y-axis is the minor axis.

$$\Rightarrow \frac{x^2}{a^2} = 1 - \frac{y^2}{b^2} \Rightarrow \frac{x^2}{a^2} = \frac{b^2 - y^2}{b^2} \Rightarrow x^2 = \frac{a^2}{b^2}(b^2 - y^2)$$



Volume of the ellipse about minor axis (Y-axis)

= Volume generated by revolution of arc B'AB about Y-axis

= 2 × volume generated by revolution of arc AB

$$= 2 \times \int_0^b \pi x^2 dy = 2\pi \times \int_0^b \frac{a^2}{b^2}(b^2 - y^2) dy$$

$$= \frac{2\pi a^2}{b^2} \int_0^b (b^2 - y^2) dy = \frac{2\pi a^2}{b^2} \left[b^2 y - \frac{y^3}{3} \right]_0^b$$

$$= \frac{2\pi a^2}{b^2} \left[b^3 - \frac{b^3}{3} \right] = \frac{2\pi a^2}{b^2} \times \frac{2b^3}{3} = \frac{4\pi a^2 b}{3} \text{ cubic unit (Ans)}$$

Ex-77: Determine the volume of the solid obtained by rotating the portion of the region bounded by $y = \sqrt[3]{x}$ and $y = \frac{x}{4}$ that lies in the first quadrant about y-axis.

Soln: $y = \sqrt[3]{x} = x^{\frac{1}{3}} \Rightarrow y^3 = x \Rightarrow x = y^3 = f(y)$ (say) —— (1)

$y = \frac{x}{4} \Rightarrow x = 4y = g(y)$ (say) —— (2)

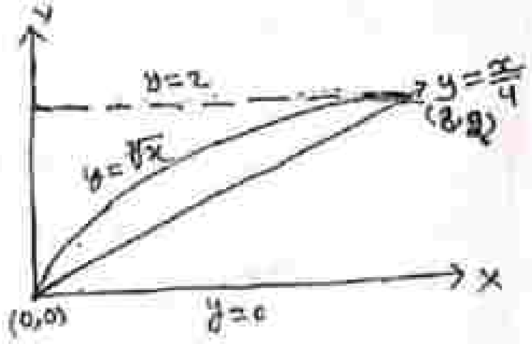
For point of intersection of ① & (2)

$$y^3 = 4y \Rightarrow y^3 - 4y = 0 \Rightarrow y(y^2-4) = 0$$

$\Rightarrow y = 0$ or $y = \pm 2 = 2$ (∵ curve lies in 1st quadrant)

$y = 0 \Rightarrow x = 0$

$y = 2 \Rightarrow x = 8$



Required volume

$$V = \int_0^2 \pi \left[g(y)^2 - f(y)^2 \right] dy$$

$$= \pi \int_0^2 \left[(4y)^2 - (y^3)^2 \right] dy = \pi \int_0^2 (16y^2 - y^6)\, dy = \pi \left[\frac{16y^3}{3} - \frac{y^7}{7} \right]_0^2$$

$$= \pi \left[\frac{128}{3} - \frac{128}{7} \right] = 128\pi \left[\frac{1}{3} - \frac{1}{7} \right] = 128\pi \times \frac{7-3}{21} = \frac{512\pi}{21} \text{ cubic unit.}$$

EXERCISE

① Find the volume of the solid generated by revolving an area bounded by the curve $y = f(x) = x^2$ on $[0, 2]$ about y-axis.

② Evaluate volume of solid generated by revolution of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about its major axis (x-axis).

③ Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 4x + 5$, $x = 1$, $x = 4$ and x-axis about x-axis

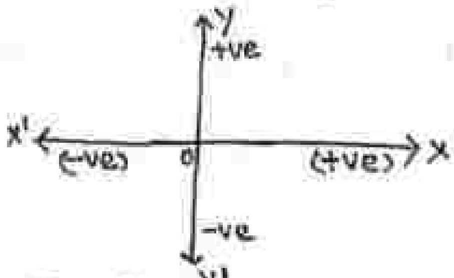
UNIT-III:
CO-ORDINATE GEOMETRY: (2D)

A: INTRODUCTION:

- The Geometry that we have studied is known as Euclidean geometry as it was the great work of Greek Mathematician Euclid.
- But in 1637 the French Mathematician Rene Descartes published his work on Co-ordinate geometry applying algebra to geometry. He is known as the father of Analytical Geometry.
- He introduced the point representation in the plane by ordered pairs of real numbers.
- There are three Co-ordinate system
  - → Cartesian Co-ordinate system.
  - → polar Co-ordinate system.
  - → oblique Co-ordinate system.

A.1 RECTANGULAR CARTESIAN CO-ORDINATE SYSTEM:

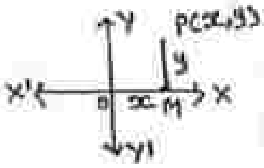
- Let's consider two mutually perpendicular lines $x'ox$ and $y'oy$ intersecting each other at the point 'o'. The point 'o' is called origin from which all measurements are done.
  The lines $x'ox$ and $y'oy$ are called x-axis and y-axis respectively. $\overrightarrow{ox}$ and $\overrightarrow{oy}$ represent the +ve direction of x-axis and y-axis respectively. $\overrightarrow{ox'}$ and $\overrightarrow{oy'}$ represents -ve direction of x-axis and y-axis respectively. This constitutes cartesian co-ordinate system.



- The branch of Mathematics in which geometrical problems are solved through algebra using co-ordinate system is called Co-ordinate geometry or Analytical geometry.

  Co-ordinate geometry relating to plane is called two-dimensional geometry (2D).

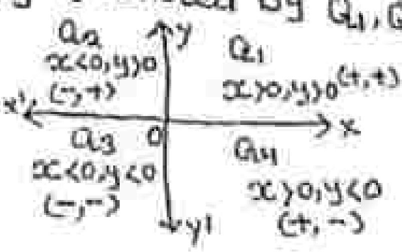
- Let 'p' be any point in the plane, $\overline{PM} \perp$ x-axis such that $OM = x$, $PM = y$. Then the order pair (x, y) is called co-ordinate of the point p.
  x = x-co-ordinate or abscissa.
  y = y-co-ordinate or ordinate.



  Also note that $x = \perp^{ar}$ distance of P from y-axis.
  $y = \perp^{ar}$ distance of P from x-axis.

- Co-ordinate of origin $= (0,0)$
  Any point on x-axis is of the form $(a, 0), a \in R$ and on y-axis is of the form $(0, a), a \in R$.

- The co-ordinate axes $x'ox$ and $y'oy$ divides the plane into four regions called Quadrants. The region xoy, $x'oy$, $x'oy'$ and xoy' are called 1st, 2nd, 3rd and fourth quadrant respectively (Denoted by Q_1, Q_2, Q_3, Q_4).
  In Q_1, $x > 0, y > 0$
  In Q_2, $x < 0, y > 0$
  In Q_3, $x < 0, y < 0$
  In Q_4, $x > 0, y < 0$



A_2: DISTANCE FORMULA:

- Distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$\boxed{AB = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}$$

- Distance of $P(x, y)$ from origin O is $\boxed{OP = \sqrt{x^2+y^2}}$

A_3: DIVISION/ SECTION FORMULA:

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two points in the plane. Then

- If P divides $\overline{AB}$ internally in the ratio m:n then co-ordinate of P is $P\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$
- If P divides $\overline{AB}$ externally in the ratio m:n then co-ordinate of P is $P\left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}\right)$.
- If P is the mid point of $\overline{AB}$ then co-ordinate of P is $P\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$
- If P divides $\overline{AB}$ in the ratio K:1 then co-ordinate of P is $\left(\frac{Kx_2+x_1}{K+1}, \frac{Ky_2+y_1}{K+1}\right)$.

If $K<0 \Rightarrow$ P divides $\overline{AB}$ externally.
If $K>0 \Rightarrow$ P divides $\overline{AB}$ internally.

A_4: Area of Triangle:

- Area of a triangle having vertices $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ is $A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)$ (Take +ve sign.)

A_5: CONDITION OF CO-LINEARITY:

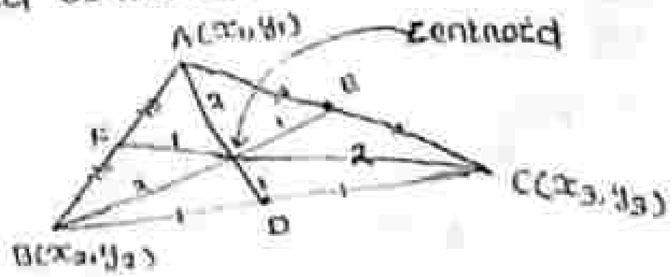
Three points $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ will be co-linear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \text{ or } x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)=0$$

A_6: CENTROID OF TRIANGLE:

- Centroid of a triangle having vertices $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

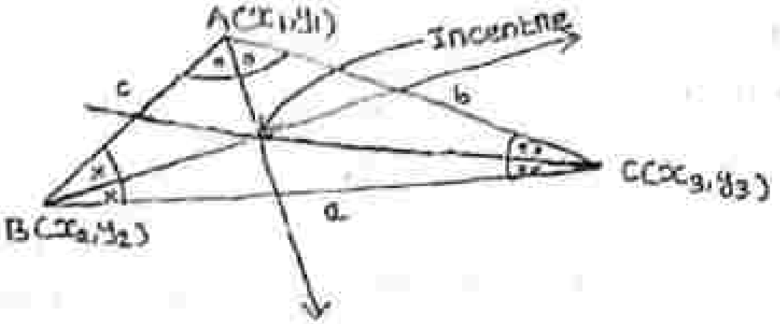
- The point at which medians of a triangle intersects one another is called centroid. Centroid divides median in the ratio 2:1



A_7: INCENTRE OF TRIANGLE:

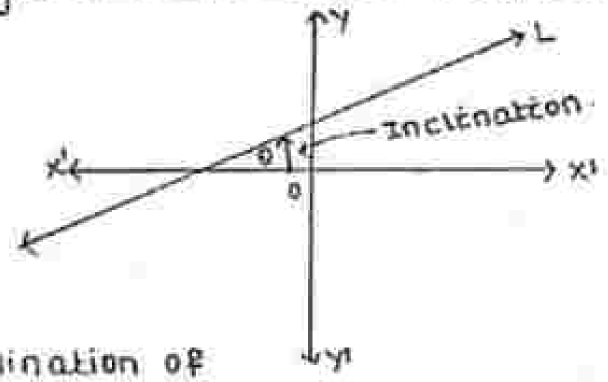
- The points at which the bisectors of angles of a triangle intersect is called Incentre
- Incentre of a triangle having vertices $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ and sides a, b, c is

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$



A_8: INCLINATION AND SLOPE OF A LINE:

- Angle of inclination of a line is the angle made by the line with +ve direction of x axis measured in anticlockwise direction.
- If θ is the angle of inclination of a line then $0° \le \theta < 180°$



- If θ is the inclination of a line then slope of the line is denoted by m and defined by

$$\boxed{m = \tan\theta}$$

- Slope of a line joining two points (x_1,y_1) and (x_2,y_2) is

$$\boxed{m = \frac{y_2-y_1}{x_2-x_1}}$$

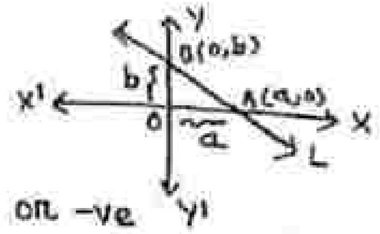
-

LINES	INCLINATION(θ)	Slope $m=\tan\theta$
X-AXIS	0°	0
Y-AXIS	90°	∞
Line parallel to X-axis (⊥er Y-axis)	0°	0
Line parallel to Y-axis ⊥er to X-axis	90°	∞

4: INTERCEPTS:

- The x-coordinate and y-co-ordinate of the points at which a line cuts x-axis and y axis respectively are called x and y intercepts respectively.

x-intercept of $L = a$
y-intercept of $L = b$



- intercept may be +ve or -ve or zero
- To find x-intercept put $y=0$
  To find y-intercept put $x=0$ in eqn of line and find x, y respectively.

Ex-1: If the distance between the point $(3,a)$ and $(6,1)$ is 5 then find a.

Soln: Given points are $A(3,a)$ and $B(6,1)$

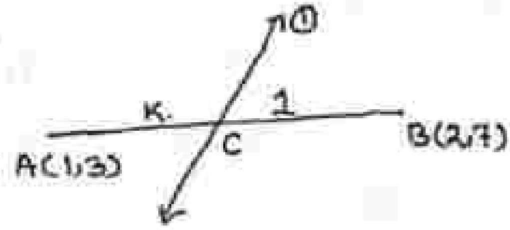
But $AB = 5 \Rightarrow \sqrt{(6-3)^2+(1-a)^2} = 5$

$\Rightarrow 9+(1-a)^2 = 25 \Rightarrow (1-a)^2 = 16 \Rightarrow 1-a = \sqrt{16} = \pm 4$

$\Rightarrow 1-a = 4$ or $1-a=-4 \Rightarrow a = 1-4$ or $a = 1+4 \Rightarrow a=-3$ or $a=5$ (Ans)

Ex-2: Find the ratio in which the line $3x+y-9=0$ divides the line segment of the points $(1,3), (2,7)$.

Soln: Given line is $3x+y-9=0$ —— ① and points are $A(1,3)$, $B(2,7)$.



Let line ① divides $\overline{AB}$ in the ratio $K:1$ at point C. Then co-ordinates of C are

$$\left(\frac{2K+1}{K+1}, \frac{7K+3}{K+1}\right)$$

But C is a point on line(1). Hence

$$3 \times \frac{2K+1}{K+1} + \frac{7K+3}{K+1} - 9 = 0 \Rightarrow \frac{6K+3}{K+1} + \frac{7K+3}{K+1} - 9 = 0$$

$$\Rightarrow \frac{6K+3+7K+3-9K-9}{K+1} = 0 \Rightarrow 4K-3=0 \Rightarrow K = \frac{3}{4}$$

$\therefore$ The required ratio is $3:4$

Ex-3: For what values of x will the points $(x,-1), (2,1)$ and $(4,5)$ lie on a line.

Soln: Given points are $A(x,-1)$, $B(2,1)$, $C(4,5)$. A, B, C will lie on a line $\Rightarrow$ A, B, C are co-linear

$$\Rightarrow \begin{vmatrix} x & -1 & 1 \\ 2 & 1 & 1 \\ 4 & 5 & 1 \end{vmatrix} = 0 \Rightarrow x(1-5)+1(2-4)+1(10-4) = 0$$

$$\Rightarrow -4x-2+6=0 \Rightarrow 4x=4 \Rightarrow x=1 \text{ (Ans)}$$

Ex-4: If area of the triangle with vertices $(0,0),(1,0),(0,a)$ is 10 unit then find a.

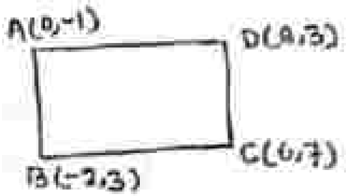
Soln: vertices of the triangle are $A(0,0)=A(x_1,y_1)$, $B(1,0)=B(x_2,y_2)$ and $C(0,a)=C(x_3,y_3)$ (say).

Area of triangle ABC = 10

$\Rightarrow \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 10 \Rightarrow \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & a & 1 \end{vmatrix} = 10$

$\Rightarrow \frac{1}{2}\left\{1\begin{vmatrix} 1 & 0 \\ 0 & a \end{vmatrix}\right\} = 10 \Rightarrow \frac{1}{2}(a-0)=10 \Rightarrow \frac{a}{2}=10 \Rightarrow a=20$ (Ans)

Ex-5: show that the points $(0,-1)$, $(-2,3)$, $(6,7)$, $(8,3)$ are vertices of a rectangle:



Soln: Given points are

$A(0,-1)$, $B(-2,3)$, $C(6,7)$, $D(8,3)$

$AB = \sqrt{(-2-0)^2+(3+1)^2} = \sqrt{20}$

$BC = \sqrt{(6+2)^2+(7-3)^2} = \sqrt{80}$

$CD = \sqrt{(8-6)^2+(3-7)^2} = \sqrt{20}$

$AD = \sqrt{(8-0)^2+(3+1)^2} = \sqrt{80}$

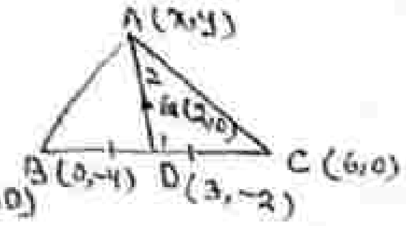
$AC = \sqrt{(6-0)^2+(7+1)^2} = \sqrt{100} = 10$

$BD = \sqrt{(8+2)^2+(3-3)^2} = 10$

$\therefore$ $AB=CD$, $AD=BC$ and $AC=BD$

$\Rightarrow$ ABCD is a rectangle.

Ex-6: Two vertices of a triangle are $(0,-4)$ and $(6,0)$. If the median meets at the point $(2,0)$ then find co-ordinate of 3rd vertex.



Soln: Two vertices of triangle are $B(0,-4)$, $C(6,0)$. Let third vertex be $A(x,y)$. D is the mid point of $\overline{BC}$, $\overline{AD}$ median. centroid $= G(2,0)$

$AG : GD = 2:1$

centroid $G = \left(\frac{x+0+6}{3}, \frac{y+0-4}{3}\right)$

$\Rightarrow (2,0) = \left(\frac{x+6}{3}, \frac{y-4}{3}\right)$

$\Rightarrow \frac{x+6}{3}=2$, $\frac{y-4}{3}=0 \Rightarrow x+6=6$, $y-4=0$

$\Rightarrow x=0, y=4$

$\therefore$ Third vertex is $(0,4)$ (Ans)

Ex-7: Find inclination of the line joining the points $(1,0), (2,\sqrt{3})$

Soln: Given points are $A(1,0), B(2,\sqrt{3})$.

slope of $\overleftrightarrow{AB}$ $m = \frac{\sqrt{3}-0}{2-1} = \frac{\sqrt{3}}{1} = \sqrt{3}$

$\Rightarrow \tan\theta = \sqrt{3}$ (θ = inclination)

$\Rightarrow \tan\theta = \tan 60^\circ \Rightarrow \theta = 60^\circ$

$\therefore$ Inclination of the line is 60°

Ex-8: If slope of the line joining the points $(3,-4)$ and $(a,2)$ is -1 then find a?

Soln: Given points are $A(3,-4), B(a,2)$.

slope of $\overleftrightarrow{AB}$,

$m = \frac{2+4}{a-3} \Rightarrow -1 = \frac{6}{a-3} \Rightarrow 6 = -a+3 \Rightarrow a = 3-6 = -3$ Ans.

EXERCISE:

1. Find the ratio in which the line segment joining $(-2,-3)$ and $(5,4)$ is divided by co-ordinate axes and hence find the co-ordinates of the points.
2. Prove that $P(3,3), Q(9,0), R(12,21)$ are the vertices of a right angled Triangle.
3. Find the co-ordinates of the points which divides $\overline{PQ}$ in the ratio 2:3 internally and externally where P, Q are $(2,-1), (-3,4)$.
4. Find Area of the Triangle having vertices $A(4,4), B(3,-2), C(-3,16)$.
5. Find the value of 'a' so that the points $(1,4), (2,7), (3,a)$ are colinear.
6. In a triangle one of the vertices is at $(2,5)$ and centroid of the triangle is at $(-1,1)$. Find the co-ordinate of mid point.
7. Show that the points $(1,1), (4,4), (4,8), (1,5)$ are vertices of a parallelogram.
8. Three vertices of a parallelogram are $(3,4), (3,8), (9,8)$. Find the fourth vertex.
9. Show that the points $(1,1), (1,-1), (-\sqrt{3},\sqrt{3})$ are the vertices of an equilateral triangle.
10. Find the ratio in which the line segment joining $(-2,-3)$ and $(5,4)$ is divided by the co-ordinate axes.
11. Find the ratio in which the line segment joining $(2,-3)$ and $(5,6)$ is divided by X-axis.
12. Find the co-ordinates of the points of trisection of the segments joining the points $(3,-8)$, and $(9,4)$.

13. Find slope of the lines whose inclinations are

(a) 30° (b) 135° (c) 120° (d) 150° (e) 45° (f) 60°

14. Find inclinations of lines having slopes

(a) $\frac{1}{\sqrt{3}}$ (b) $\sqrt{3}$ (c) $-\sqrt{3}$ (d) 1 (e) $-\frac{1}{\sqrt{3}}$

15. Find incination of the line joining the points (2,3) ar (3,4)

16. If slope of the line joing the points (a,2) and (3,4) is $\frac{1}{\sqrt{3}}$ then find a. Hence find its inclination.

B: EQUATION OF LINE

Equation of line is a relation in x and y to which all point on the line satisfies such that x and y are of first degree

B_1: GENERAL FORM: Every 1st. degree Equation in x and y will represent a straight line.

Thus the general form of line is

$\boxed{ax+by+c=0}$ where a and b are not simultaneously zero.

B_2: SLOPE-INTERCEPTS:

$\boxed{y = mx+c}$ where m = slope, c = y-intercept.

B_3: SLOPE-POINT FORM:

Equation of line passing through the point (x_1,y_1) having slope m is $\boxed{y-y_1 = m(x-x_1)}$

B_4: TWO POINT FORM:

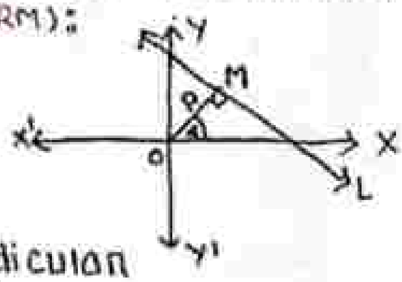
Equation of line passing through the points (x_1,y_1) and (x_2,y_2) is $\boxed{y-y_1 = \frac{y_2-y_1}{x_2-x_1}(x-x_1)}$

B_5: INTERCEPT FORM:

$\boxed{\frac{x}{a}+\frac{y}{b}=1}$ where a = x-intercept
b = y-intercept

B_6: NORMAL FORM (PERPENDICULAR FORM):

$\boxed{x\cos\alpha + y\sin\alpha = p}$



Where p = Length of $\perp^{ar}$ from origin to line.

α = Angle made by that perpendicular with positive direction of x-axis.

B4: VERTICAL AND HORIZONTAL LINES:

LINES:	
x-axis	$y=0$
Line parallel to x-axis or perpendicular to y-axis	$y=K$ (constant)
y-axis	$x=0$
Line parallel to y-axis or $\perp^{ar}$ to x-axis	$x=K$ (constant)

C: CONVERSION OF GENERAL FORM OF LINE INTO:

C1: SLOPE-INTERCEPT FORM:

$ax+by+c=0 \Rightarrow by=-ax-c \Rightarrow y=\frac{-ax-c}{b}$

$\Rightarrow y=\left(-\frac{a}{b}\right)x+\left(-\frac{c}{b}\right)$ which is in the form of $y=mx+c$

where $\boxed{\text{slope } m=-\frac{a}{b}}$, $\boxed{y\text{-intercept}(c)=-\frac{c}{b}}$

C2: INTERCEPT FORM:

$ax+by+c=0 \Rightarrow ax+by=-c \Rightarrow \frac{ax+by}{-c}=1$

$\Rightarrow \frac{ax}{-c}+\frac{by}{-c}=1 \Rightarrow \frac{x}{(-c/a)}+\frac{y}{(-c/b)}=1$ which is in the form of $\frac{x}{a}+\frac{y}{b}=1$ where $\boxed{x\text{ intercept}=-\frac{c}{a}}$, $\boxed{y\text{-intercept}=-\frac{c}{b}}$.

C3: NORMAL FORM:

$ax+by+c=0$

$\Rightarrow ax+by=-c$ (Take the constant term on R.H.S.)

$\Rightarrow -ax-by=c$ (make $-c$ positive if $-c<0$)

$\Rightarrow \frac{-ax}{\sqrt{a^2+b^2}}-\frac{by}{\sqrt{a^2+b^2}}=\frac{c}{\sqrt{a^2+b^2}}$ (Divide both sides by $\sqrt{a^2+b^2}$)

. Which is in the form of $x\cos\alpha+y\sin\alpha=p$ where $\cos\alpha=\frac{-a}{\sqrt{a^2+b^2}}$, $\sin\alpha=\frac{-b}{\sqrt{a^2+b^2}}$, $p=\frac{c}{\sqrt{a^2+b^2}}$

D: POINT OF INTERSECTION OF TWO LINES:

To find point of intersection of two lines $a_1x+b_1y+c_1=0$ —— (1) and $a_2x+b_2y+c_2=0$ —— (2)

- Solve for x and y (Eqn (1) and Eqn (2)) then (x,y) is the point of intersection.

 both

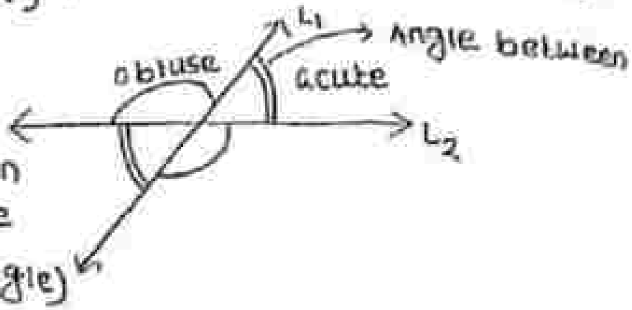
- Find x and y using the formula $x=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}$

$y=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}$

E: ANGLE BETWEEN TWO LINES:

- There are two angles generated between two straight lines i.e. acute angle and obtuse angle (If they are not perpendicular to each other)

- Usually the acute is taken as the angle between two lines (If the lines are perpendicular, it is right angle)



- If m_1 and m_2 are slopes of two lines and θ is the angle between them then

$$\boxed{\theta = \tan^{-1}\left|\frac{m_1-m_2}{1+m_1m_2}\right|}$$

- If $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ and θ is the angle between them then

$$\boxed{\theta = \tan^{-1}\left|\frac{a_2b_1-a_1b_2}{a_1a_2+b_1b_2}\right|}$$

- Above two lines are

Perpendicular $\Leftrightarrow$	$a_1a_2+b_1b_2=0$
Parallel $\Leftrightarrow$	$\frac{a_1}{a_2}=\frac{b_1}{b_2}$
Identical (concident) $\Leftrightarrow$	$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

F: PARALLEL AND PERPENDICULAR LINES:

- Equation of line parallel to the line $ax+by+c=0$ is $\boxed{ax+by+K=0}$ where K is any constant, $K\neq c$
- Equation of any line perpendicular to $ax+by+c=0$ is $\boxed{bx-ay+K=0}$

G: EQUATION OF LINE PASSING THROUGH POINT OF INTERSECTION OF TWO LINES.

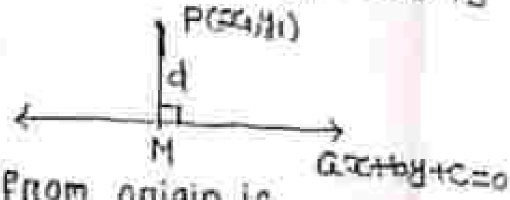
Equation of line passing through point of intersection of two lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ is

$$\boxed{(a_1x+b_1y+c_1)+K(a_2x+b_2y+c_2)=0}$$

DISTANCE OF A POINT FROM A LINE:

- Distance of the point (x_1, y_1) from the line $ax+by+c=0$ is

$$\boxed{d = \left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|}$$



- Distance of the line $ax+by+c=0$ from origin is

$$\boxed{d = \left|\frac{c}{\sqrt{a^2+b^2}}\right|}$$

- Distance between two parallel lines $ax+by+c_1=0$ and $ax+by+c_2=0$ is

$$\boxed{d = \left|\frac{c_1-c_2}{\sqrt{a^2+b^2}}\right|}$$

Ex-9: Find Equation of line which passes through the point (3,4) and sum of its intercepts is 14.

Soln: Let Equation of Required line be $\frac{x}{a}+\frac{y}{b}=1$ —— ①

Line ① passes through the point (3,4)

$\Rightarrow \frac{3}{a}+\frac{4}{b}=1$ —— ②

But $a+b=14 \Rightarrow b=14-a$ —— (3)

From ② $\frac{3}{a}+\frac{4}{b}=1 \Rightarrow \frac{3}{a}+\frac{4}{14-a}=0 \Rightarrow \frac{42-3a+4a}{a(14-a)}=1$

$\Rightarrow \frac{42+a}{14a-a^2}=1 \Rightarrow 42+a=14a-a^2 \Rightarrow a^2-13a+42=0 \Rightarrow a^2-6a-7a+42=0$

$\Rightarrow a(a-6)-7(a-6)=0 \Rightarrow (a-6)(a-7)=0 \Rightarrow a-6=0$ or $a-7=0$

$\Rightarrow a=6$ or $a=7$.

If $a=6 \Rightarrow b=14-a=14-6=8$

$\therefore$ Required line is $\frac{x}{a}+\frac{y}{b}=1 \Rightarrow \frac{x}{6}+\frac{y}{8}=1 \Rightarrow 8x+6y-48=0$

$\Rightarrow 4x+3y-24=0$ —— ④

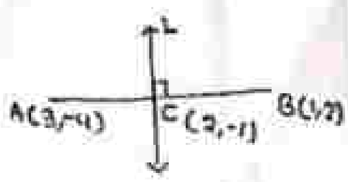
If $a=7 \Rightarrow b=14-a=14-7=7$

Required line is $\frac{x}{a}+\frac{y}{b}=1 \Rightarrow \frac{x}{7}+\frac{y}{7}=1 \Rightarrow x+y-7=0$ —(5)

$\therefore$ Equation of required line is $4x+3y-24=0$ or $x+y-7=0$ (ANS)

Ex-10: Find Equation of line bisecting the line segment joining (3,-4) and (1,2) at right angle.

Given points are A(3,-4) and B(1,2). Line L Bisects $\overline{AB}$ at C. Hence 'C' is the mid point of $\overline{AB}$.



$\therefore$ Co-ordinate of C $=\left(\frac{3+1}{2}, \frac{-4+2}{2}\right)=(2,-1)$

Slope of $\overline{AB}$ $(m_1) = \frac{2-(-4)}{1-3} = \frac{6}{-2} = -3$

But $L \perp \overline{AB} \Rightarrow$ slope of L $(m_2) = -\frac{1}{m_1} = \frac{1}{3}$

Now L passes through the point $C(2,-1) = C(x_1, y_1)$ say.

$\therefore$ Eq^n of L is $y - y_1 = m_2(x - x_1)$

$$\Rightarrow y-(-1) = \frac{1}{3}(x-2) \Rightarrow 3y+3 = x-2$$

$\Rightarrow x - 3y - 2 - 3 = 0 \Rightarrow x - 3y - 5 = 0$

$\therefore$ Equation of required line is $x - 3y - 5 = 0$ (Ans)

Ex-11: Find Equation of line passing through $(1,-2)$ and making intercept 2 on y-axis.

Sol^n: Let Equation of required line be $y = mx + c$ — ① where m = slope and c = y-intercept.

But it is given that $c = 2$

Also the line ① passes through $(1,-2)$

$\therefore -2 = m \times 1 + 2 \Rightarrow m = -4$

Hence equation of required line is

$y = mx + c \Rightarrow y = -4x + 2 \Rightarrow 4x + y - 2 = 0$ (Ans).

Ex-12: Find Equation of line passing through the point $(2,3)$ and parallel to the join of $(4,-5)$ and $(-7,3)$.

Sol^n: Given points are $A(2,3), B(4,-5), C(-7,3)$

Slope of $\overline{BC}$ $(m_1) = \frac{3-(-5)}{-7-4} = \frac{8}{-11} = -\frac{8}{11}$

Since required line is parallel to $\overline{BC}$,

slope of required line $m = m_1 = -\frac{8}{11}$

Again required line passes through $(2,3) = (x_1, y_1)$ say

Hence its equation is $y - y_1 = m(x - x_1)$

$$\Rightarrow y - 3 = \frac{-8}{11}(x-2)$$

$\Rightarrow 11y - 33 = -8x + 16 \Rightarrow 8x + 11y - 33 - 16 = 0 \Rightarrow 8x + 11y - 49 = 0$ (Ans)

Ex-13: Reduce $2x - 3y + 7 = 0$ to the slope form and find its intercepts on y-axis.

Sol^n: Given line is $2x - 3y + 7 = 0 \Rightarrow -3y = -2x - 7$

$$\Rightarrow y = \frac{-2x-7}{-3} = \frac{2x}{3} + \frac{7}{3} = \left(\frac{2}{3}\right)x + \left(\frac{7}{3}\right)$$

which is in the form of $y = mx + c$ (slope form)

Intercept on y-axis is $\frac{7}{3}$ (Ans)

Ex-14: Reduce $3x + 5y + 4 = 0$ to intercept form and hence find its intercept.

Sol^n: Given line is $3x + 5y + 4 = 0 \Rightarrow 3x + 5y = -4$

$$\Rightarrow \frac{3x+5y}{-4} = 1 \Rightarrow \frac{3x}{-4} + \frac{5y}{-4} = 1 \Rightarrow \frac{x}{(-4/3)} + \frac{y}{(-4/5)} = 1$$

which is the required intercept form. x-intercept $= -\frac{y}{3}$ and y-intercept $= \frac{-y}{3}$ (Ans)

15: Reduce $3x-y+2=0$ to normal form and find the length of the perpendicular from origin to the line.

Soln: $3x-y+2=0 \Rightarrow 3x-y=-2 \Rightarrow -3x+y=2$

$\Rightarrow \frac{-3x+y}{\sqrt{(3)^2+(-1)^2}} = \frac{2}{\sqrt{(3)^2+(-1)^2}} \Rightarrow \frac{-3x+y}{\sqrt{10}} = \frac{2}{\sqrt{10}}$

$\Rightarrow \frac{-3x}{\sqrt{10}} + \frac{y}{\sqrt{10}} = \frac{2}{\sqrt{10}} \Rightarrow \left(\frac{-3}{\sqrt{10}}\right)x + \left(\frac{1}{\sqrt{10}}\right)y = \frac{2}{\sqrt{10}}$ which is the required Normal form $x\cos\alpha + y\sin\alpha = p$ where $\cos\alpha = \frac{-3}{\sqrt{10}}$, $\sin\alpha = \frac{1}{\sqrt{10}}$, $p = \frac{2}{\sqrt{10}}$

Length of the perpendicular from origin is $\frac{2}{\sqrt{10}}$ unit.

16: Find Equation of line parallel to x-axis and passing through (1,3)

Soln: Equation of line parallel to x-axis is $y=k$ —— ①

Line ① passes through (1,3). Hence $3=k \Rightarrow k=3$

$\therefore$ Required line is $y=k \Rightarrow y=3 \Rightarrow y-3=0$ (Ans)

17. Find Equation of line passing through $(-4,2)$ and parallel to the line $4x-3y=0$.

Soln: Given point is $P(-4,2)$ and line is $4x-3y=0$ —— ①

Let required line be L

Since L ∥ Line ① $\Rightarrow$ Equation of L is $4x-3y+k=0$ —— ②

L passes through $P(-4,2) \Rightarrow 4(-4)-3(2)+k=0$

$\Rightarrow -22+k=0 \Rightarrow k=22$.

$\therefore$ Eqn of L is $4x-3y+k=0 \Rightarrow 4x-3y+22=0$ (Ans)

18. Find Equation of line passing through the point $(-2,3)$ and perpendicular to the line $3x+4y-11=0$

Soln: Given point is $P(-2,3)$ and line is $3x+4y-11=0$ —— ①

Let required line be L.

L ⊥ Line ① $\Rightarrow$ Equation of L is $4x-3y+k=0$ —— ②

But L passes through $P(-2,3)$

$\Rightarrow 4(-2)-3(3)+k=0 \Rightarrow -17+k=0 \Rightarrow k=17$

$\therefore$ Equation of L is $4x-3y+k=0 \Rightarrow 4x-3y+17=0$ (Ans)

19. Find Equation of line passing through the point of intersection of of lines $x+3y+2=0$ and $x-2y-4=0$ and perpendicular to the line $2y+5x-9=0$.

Soln: Given lines are $x+3y+2=0$ —— ①

$x-2y-4=0$ —— ②

and $2y+5x-9=0 \Rightarrow 5x+2y-9=0$ —— ③

Point of intersection of line ① and ② is

$$P\left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}, \frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right) = P\left(\frac{(3)(-4)-(-2)(2)}{(1)(-2)-(1)(3)}, \frac{(2)(1)-(-4)(1)}{(1)(-2)-(1)(3)}\right)$$

$= P\left(\frac{-8}{-5}, \frac{6}{-5}\right) = P\left(\frac{8}{5}, -\frac{6}{5}\right)$

Required line L is perpendicular to line ③

Hence Equation of L is

$$2x - 5y + K = 0 \quad \text{——④}$$

L passes through $P\left(\frac{8}{5}, \frac{-6}{5}\right)$

$\therefore 2 \times \frac{8}{5} - 5 \times \left(\frac{-6}{5}\right) + k = 0$

$\Rightarrow \frac{16}{5} + 6 + k = 0$

$\Rightarrow k = \frac{-16}{5} - 6 = \frac{-16 - 30}{5} = \frac{-46}{5}$

Equation of L is

$$2x - 5y + k = 0$$

$\Rightarrow 2x - 5y - \frac{46}{5} = 0$

$\Rightarrow \frac{10x - 25y - 46}{5} = 0$

$\Rightarrow 10x - 25y - 46 = 0$ (Ans)

20: Find the length of the perpendicular drawn from the point $(-3, -4)$ to the line $12x - 5y + 65 = 0$

Sol^n: Given line is $12x - 5y + 65 = 0$ ——①

Given point is $P(-3, -4)$

Length of the Perpendicular

$$= \left|\frac{12(-3) - 5(-4) + 65}{\sqrt{(12)^2 + (-5)^2}}\right| = \left|\frac{-36 + 20 + 65}{\sqrt{144 + 25}}\right|$$

$$= \left|\frac{49}{13}\right| = \frac{49}{13} \text{ unit.}$$

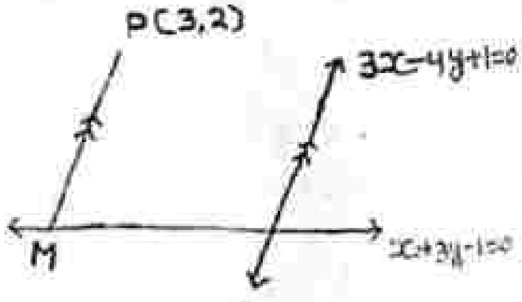
21: Find the distance of the point $(3,2)$ from the line $x + 3y - 1 = 0$ measured parallel to the line $3x - 4y + 1 = 0$.

Sol^n: Given point is $P(3,2)$ and

Lines are

$$x + 3y - 1 = 0 \quad \text{——①}$$

$$3x - 4y + 1 = 0 \quad \text{——②}$$



M be any point on Line ①

such that $\overline{PM} \parallel$ Line (2)

We have to find out PM.

$\overline{PM} \parallel$ Line (2)

Hence Equation of PM is $3x - 4y + K = 0$ ——(3)

But $\overline{PM}$ passes through $P(3,2)$

$3\times3-4\times2+K=0 \Rightarrow 9-8+K=0 \Rightarrow K+1=0 \Rightarrow K=-1$

Equation of $\overleftrightarrow{PM}$ is $3x-4y+K=0 \Rightarrow 3x-4y-1=0$ —— (4)

M is the point of intersection of Line (1) and (4)

Co-ordinates of $M = M\left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}, \frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right)$

$= M\left(\frac{3\times(-1)-(-4)(-1)}{(1)(-4)-(3)(3)}, \frac{(-1)(3)-(-1)(1)}{(1)(-4)-(3)(3)}\right) = M\left(\frac{7}{13}, \frac{2}{13}\right)$

Now $PM = \sqrt{\left(\frac{7}{13}-3\right)^2+\left(\frac{2}{13}-2\right)^2} = \sqrt{\left(\frac{-32}{13}\right)^2+\left(\frac{-24}{13}\right)^2} = \sqrt{\frac{1024+576}{13^2}} = \sqrt{\frac{1600}{13^2}} = \frac{40}{13}$

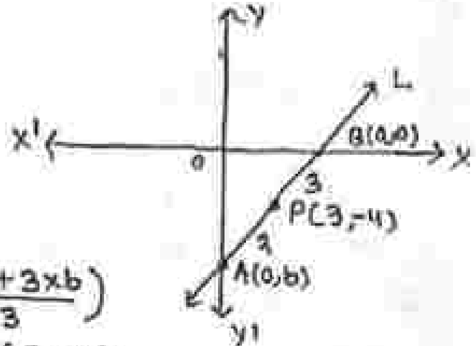
∴ required distance is $\frac{40}{13}$ unit (Ans).

23: Find Equation of Line which passes through the point $(3,-4)$ and is such that its portion between the axes is divided internally in the ratio $2:3$ at that point.

Soln: Given point is $P(3,-4)$

Let Equation of required Line L be

$\frac{x}{a}+\frac{y}{b}=1$ —— (1)



∴ L cuts x axis at $B(a,0)$ and y-axis at $A(0,b)$. $AP:PB = 2:3$

∴ Co-ordinates of $P = P\left(\frac{2\times a+3\times0}{2+3}, \frac{2\times0+3\times b}{2+3}\right)$

$= P\left(\frac{2a}{5}, \frac{3b}{5}\right) = P(3,-4)$

$\Rightarrow \frac{2a}{5}=3$ and $\frac{3b}{5}=-4$

$\Rightarrow a=\frac{15}{2}$ and $b=-\frac{20}{3}$

Eqn of L is $\frac{x}{a}+\frac{y}{b}=1 \Rightarrow \frac{x}{15/2}+\frac{y}{-20/3}=1 \Rightarrow \frac{2x}{15}-\frac{3y}{20}=1$

$\Rightarrow \frac{2x}{15}\times60-\frac{3y}{20}\times60=60 \Rightarrow 8x-9y-60=0$ (ANS).

24: Find the foot of the perpendicular from the point $(2,3)$ on the Line $3x-4y+7=0$

Soln: Given point is $P(2,3)$ and Line is

$3x-4y+7=0$ —— (1)

$\overline{PM} \perp$ Line (1), M is the foot of perpendicular.

∵ $\overline{PM} \perp$ Line (1), Equation of $\overleftrightarrow{PM}$ is $-4x-3y+K=0$ —— (2)

But $P(2,3)$ is a point on Line (2)

∴ $-4\times2-3\times3+K=0 \Rightarrow K=17$

∴ Eqn of $\overleftrightarrow{PM}$ is $-4x-3y+K=0 \Rightarrow -4x-3y+17=0 \Rightarrow 4x+3y-17=0$ —— (3)

Eqn (1) × 4 $\Rightarrow 12x-16y+28=0$

Eqn (3) × 3 $\Rightarrow \underset{(-)}{12x}\underset{(-)}{+9y}\underset{(+)}{-51}=0$

$-25y+79=0 \Rightarrow y=\frac{79}{25}$

From (1), $3x-4\times\frac{79}{25}+7=0 \Rightarrow 3x=\frac{316}{25}-7=\frac{316-175}{25}=\frac{141}{25}$

$\Rightarrow x=\frac{141}{25}\times\frac{1}{3}=\frac{47}{25}$

Since M is the point of intersection of Line(1) and (3),
Co-ordinates of M $= M(x,y) = M\left(\frac{47}{25}, \frac{79}{25}\right)$

$\therefore$ Required foot of the perpendicular is $\left(\frac{47}{25}, \frac{79}{25}\right)$ (ANS)

25: Find K if $2x-3y+4=0$ and $4x+Ky+1=0$ are parallel.

Soln: Given Lines are $2x-3y+4=0$ —— ①
and $4x+Ky+1=0$ —— ②

Line① ∥ Line② $\Rightarrow \frac{2}{4} = \frac{-3}{K} \Rightarrow 2K=-12 \Rightarrow K=-6$ (ANS)

26: Find K if $2x+3y-1=0$ and $Kx-4y+2=0$ are perpendicular

Soln: Given Lines are $2x+3y-1=0$ —— ①
$Kx-4y+2=0$ —— ②

Line① ⊥ Line② $\Rightarrow (2)(K)+(3)(-4)=0 \Rightarrow 2K=12 \Rightarrow K=6$ (ANS)

27: Find K if $2x+3y+K=0$ and $Kx-3y+2=0$ are identical.

Soln: Given Lines are $2x+3y+K=0$ —— ①
$Kx-3y+2=0$ —— ②

Line ① and ② are identical

$$\frac{2}{K} = \frac{3}{-3} = \frac{K}{2} \Rightarrow K^2=4 \Rightarrow K=\sqrt{4}=\pm 2$$

But if $K=2$ then the above equality will not be meaningful.
Hence $K=-2$. (ANS)

28: Find the distance between the Lines $2y-3=0$ and $3y-2=0$

Soln: Given Lines are $2y-3=0$ and $3y-2=0$

$\Rightarrow y-\frac{3}{2}=0$ —— ①
$y-\frac{2}{3}=0$ —— ②

$\therefore$ Line① ∥ Line②

$\Rightarrow$ Required distance $= \left|\frac{-\frac{3}{2}-\left(-\frac{2}{3}\right)}{\sqrt{0^2+1^2}}\right| = \left|-\frac{3}{2}+\frac{2}{3}\right| = \left|\frac{-9+4}{6}\right|$

$$= \left|-\frac{5}{6}\right| = \frac{5}{6} \text{ unit (Ans)}$$

29. Find the angle between the Lines $x+y+7=0$ and $x-y+1=0$

Soln: Given Lines are $x+y+7=0$ —— ①
$x-y+1=0$ —— ②

Here $a_1=1,\ b_1=1,\ c_1=7$
$a_2=1,\ b_2=-1,\ c_2=1$

Let θ be the Angle between Line① & ②,

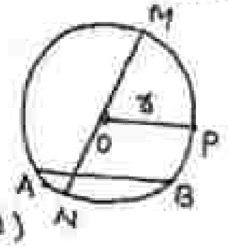
$$\theta = \tan^{-1}\left|\frac{a_2b_1-a_1b_2}{a_1a_2+b_1b_2}\right| = \tan^{-1}\left|\frac{(1)(1)-(1)(-1)}{(1)(1)+(1)(-1)}\right|$$

$$= \tan^{-1}\left|\frac{2}{0}\right| = \tan^{-1}\infty = \frac{\pi}{2}$$

$\therefore$ Angle between the Lines is $\frac{\pi}{2}$.

CIRCLE:

- The Locus of a point which moves in a plane such that its distance from a fixed point is always constant is called a circle.
- The fixed point is called centre and fixed distance is called radius. O = centre, $OP = r$ = radius.
- The length of boundary of the circle is called circumference.



- The line segment joining any two points on the circle is called chord ($\overline{AB}, \overline{MN}$)
- Any chord passing through centre of the circle is called diameter. MN = Diameter $= d$
- Note that $\boxed{d = 2r}$

$\boxed{\text{Circumference} = 2\pi r}$ $\pi = \frac{22}{7}$

$\boxed{\text{Area of the circle} = \pi r^2}$

- Two circles having same centre are called concentric circles.

I: EQUATION OF CIRCLE:

I₁: STANDARD FORM (CENTRE-RADIUS FORM):

- Equation of circle having centre at (h,k) and radius r is $\boxed{(x-h)^2+(y-k)^2 = r^2}$

I₂: EQUATION OF CIRCLE WITH ENDS OF A DIAMETER:

Equation of circle having end points (x_1,y_1) and (x_2,y_2) of a diameter is

$$\boxed{(x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0}$$

I₃: GENERAL FORM:

- $\boxed{x^2+y^2+2gx+2fy+c=0}$ is the general form of equation of circle where $g^2+f^2 \geq c$
- $\boxed{\text{Centre} = (-g, -f), \text{ radius } r = \sqrt{g^2+f^2-c}}$
- Equation of circle is a
  - 2nd degree equation in x and y.
  - It contains no terms of xy
  - Co-efficient of x^2 = co-efficient of y^2.
  - $ax^2+2hxy+by^2+2gx+2fy+c=0$ will represent a circle if $\boxed{h=0,\ a=b\neq 0}$
- If $g^2+f^2 = c \Rightarrow r=0$ then the circle is called a point/zero circle (Real)

If $g^2+f^2 > c \Rightarrow r$ is real and circle is Real circle

If $g^2+f^2 < c \Rightarrow r$ is imaginary and the circle is imaginary.

I_7: TO FIND Equation of circle PASSING THROUGH THREE NON-COLLINEAR POINTS:

STEP-i

Given points are $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$

STEP-ii:

Let Equation of circle passing through A,B,C be

$$x^2+y^2+2gx+2fy+c=0 \quad \text{---} (1)$$

STEP-iii

Since circle(1) passes through A, B, C.

$$x_1^2+y_1^2+2gx_1+2fy_1+c=0 \quad \text{---} (2)$$
$$x_2^2+y_2^2+2gx_2+2fy_2+c=0 \quad \text{---} (3)$$
$$x_3^2+y_3^2+2gx_3+2fy_3+c=0 \quad \text{---} (4)$$

STEP-IV: Solve Eqn(2),(3) and(4) for g, f, c

STEP-V: put the values of g, f, c in eqn(1) and get the circle.

Ex 30: Find centre and radius of the circle of the circle $2x^2+2y^2-5x+3y-11=0$

Soln: $2x^2+2y^2-5x+3y-11=0$

$$\Rightarrow x^2+y^2-\frac{5}{2}x+\frac{3}{2}y-\frac{11}{2}=0 \quad \text{---} (1)$$

Comparing Eqn(1) with general form,

$$2g=-\frac{5}{2} \Rightarrow g=-\frac{5}{4}$$
$$2f=\frac{3}{2} \Rightarrow f=\frac{3}{4} \text{ and } c=-\frac{11}{2}$$

Centre $=(-g,-f)=\left(\frac{5}{4},-\frac{3}{4}\right)$

Radius $r=\sqrt{g^2+f^2-c}$

$$=\sqrt{\left(-\frac{5}{4}\right)^2+\left(\frac{3}{4}\right)^2+\frac{11}{2}}=\sqrt{\frac{25}{16}+\frac{9}{16}+\frac{11}{2}}=\sqrt{\frac{25+9+88}{16}}=\frac{\sqrt{122}}{4}$$

$\therefore$ Centre $=\left(\frac{5}{4},-\frac{3}{4}\right)$, Radius $=\frac{\sqrt{122}}{4}$ (ANS)

31: Find Equation of circle whose centre is at (1,4) and which passes through (-2,1)

Soln: Centre of the circle is at $C(1,4)=C(h,k)$ (say)

Circle passes through the point $P(-2,1)$

$\therefore$ Radius $r=CP=\sqrt{(-2-1)^2+(1-4)^2}=\sqrt{18}$

Equation of required circle is

$$(x-h)^2+(y-k)^2=r^2$$
$$\Rightarrow (x-1)^2+(y-4)^2=(\sqrt{18})^2$$
$$\Rightarrow x^2-2x+1+y^2-8y+16=18$$
$$\Rightarrow x^2+y^2-2x-8y-1=0 \text{ (ANS)}.$$

Ex: Find Equation of circle concentric with the circle $x^2+y^2-8x+6y-5=0$ and passing through the point $(-2,-7)$.

Sol^n: Given circle is $x^2+y^2-8x+6y-5=0$ —— ①

$\Rightarrow x^2-8x+y^2+6y = 5$

$\Rightarrow (x)^2-2(x)(4)+(4)^2+(y)^2+2(y)(3)+3^2 = 5+4^2+3^2$

$\Rightarrow (x-4)^2+(y+3)^2 = 30 = (\sqrt{30})^2$

$\therefore$ centre of circle ① is $(4,-3)$.

Since Required circle is concentric with circle ①,

centre of required circle $= C(4,-3) = C(h,k)$ (say)

Again required circle passes through $P(-2,-7)$

$\therefore$ radius $r = CP = \sqrt{(-2-4)^2+(-7+3)^2} = \sqrt{36+16} = \sqrt{52}$

Equation of required circle is

$$(x-h)^2+(y-k)^2=r^2 \Rightarrow (x-4)^2+(y+3)^2 = (\sqrt{52})^2$$

$\Rightarrow x^2-8x+16+y^2+6y+9 = 52$

$\Rightarrow x^2+y^2-8x+6y-27=0$ (ANS)

Ex33: Find Equation of circle whose ends of a diameter are the point of intersections of the lines $x+y-1=0$, $4x+3y+1=0$, and $4x+y+3=0$, $x-2y+3=0$

Sol^n: Given lines are

$x+y-1=0$ —— ①
$4x+3y+1=0$ —— ②
$4x+y+3=0$ —— ③
$x-2y+3=0$ —— ④

Solving Equation ① and ② by cross multiplication,

$$\frac{x}{(1)(1)-(3)(-1)} = \frac{y}{(-1)(4)-(1)(1)} = \frac{1}{(1)(3)-(4)(1)}$$

$\Rightarrow \frac{x}{4} = \frac{y}{-5} = \frac{1}{-1} \Rightarrow x = \frac{4}{-1} = -4,\ y = \frac{-5}{-1} = 5$

$\therefore$ point of intersection of line ① and ② $= (x,y) = (-4,5)$

Hence one End of a diameter of the circle $= (-4,5) = (x_1,y_1)$ (say)

Again solving ③ and ④

$$\frac{x}{(1)(3)-(-2)(3)} = \frac{y}{(3)(1)-(4)(3)} = \frac{1}{(4)(-2)-(1)(1)}$$

$\Rightarrow \frac{x}{9} = \frac{y}{-9} = \frac{1}{-9} \Rightarrow x = \frac{9}{-9} = -1,\ y = \frac{-9}{-9} = 1$

$\therefore$ Point of intersection of line ③ and ④ $= (x,y) = (-1,1)$

Hence other end of the diameter of circle $= (-1,1) = (x_2, y_2)$ (say)

$\therefore$ Equation of required circle is

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

$\Rightarrow (x+4)(x+1)+(y-5)(y-1)=0$

$\Rightarrow x^2+x+4x+4+y^2-y-5y+5=0$

$\Rightarrow x^2+y^2+5x-6y+9=0$ (ANS)

34: Find equation of circle whose centre is on the line $8x+5y=0$ and passing through the points $(2,1),(3,5)$

Sol^n: Given line is $8x+5y=0$ —— ①

and points are $A(2,1), B(3,5)$

Let Equation of required circle be

$x^2+y^2+2gx+2fy+c=0$ —— ②

Centre of the circle ② is $(-g,-f)$ lies on line ①

$\therefore$ $-8g-5f=0 \Rightarrow 8g+5f=0$ —— ③

circle ② passes through the points $A(2,1), B(3,5)$

$\therefore$ $2^2+1^2+2g\times2+2f\times1+c=0$

and $3^2+5^2+2g\times3+2f\times5+c=0$

$\Rightarrow$ $5+4g+2f+c=0$ —— ④

$34+6g+10f+c=0$ —— ⑤

Eq^{n}⑤ - Eq^n(4) $\Rightarrow 2g+8f=-29$ —— ⑥

Equation ③$\times 1 \Rightarrow 8g+5f=0$

Eq^n (6)$\times 4 \Rightarrow 8g+32f=-116$

$\quad(-)\quad(-)\qquad(+)$

$\Rightarrow -27f=116$

$\Rightarrow \boxed{f=-\frac{116}{27}}$

From Eq^{n}③

$8g+5f=0 \Rightarrow 8g+5\left(\frac{-116}{27}\right)=0$

$\Rightarrow 8g=\frac{580}{27} \Rightarrow g=\frac{580}{27}\times\frac{1}{8}=\frac{145}{54}$

$\boxed{g=\frac{145}{54}}$

From Equation ④ $5+4g+2f+c=0$

$\Rightarrow c=-5-4g-2f=-5-4\times\frac{145}{54}+2\times\frac{116}{27}$

$=-5-\frac{290}{27}+\frac{232}{27}=\frac{-135-290+232}{27}$

$\Rightarrow \boxed{c = -\frac{193}{27}}$

$\therefore$ Equation of required circle is

$$x^2+y^2+2gx+2fy+c=0$$

$$\Rightarrow x^2+y^2+2\left(\frac{145}{54}\right)x+2\left(-\frac{116}{27}\right)y-\frac{193}{27}=0$$

$$\Rightarrow 27x^2+27y^2+145x-232y-193=0 \quad \text{(Ans)}$$

Ex 35: Find Equation of circle which passes through origin and cut-off intercepts a and b from the axes.

Solⁿ: Let Equation of required circle be $x^2+y^2+2gx+2fy+c=0$ —— ①

circle ① passes through origin i.e (0,0)

$\therefore \quad 0^2+0^2+2g\times 0+2f\times 0+c=0 \Rightarrow \boxed{c=0}$

Again circle ① cut off intercepts a and b from x-axis and y axis i.e it passes through (a,0) and (0,b).

$\therefore \quad a^2+0^2+2ga+2f\times 0+0=0$ and $0^2+b^2+2g\times 0+2f\times b+0=0$

$\Rightarrow \quad a^2+2ga=0$ and $b^2+2fb=0$

$\Rightarrow \quad a(a+2g)=0$ and $b(b+2f)=0$

$\Rightarrow \quad a+2g=0$ and $b+2f=0 \quad \because a\neq 0, b\neq 0$

$\Rightarrow \quad \boxed{g=-\frac{a}{2}}$ and $\boxed{f=-\frac{b}{2}}$

Equation of circle is $x^2+y^2+2gx+2fy+c=0$

$$\Rightarrow x^2+y^2+2\left(-\frac{a}{2}\right)x+2\left(-\frac{b}{2}\right)y+0=0$$

$$\Rightarrow x^2+y^2-ax-by=0 \quad \text{(Ans)}$$

Ex 36: For what values of k the equation $2x^2-(k-1)y^2-6x+4y-1=0$ will represent a circle.

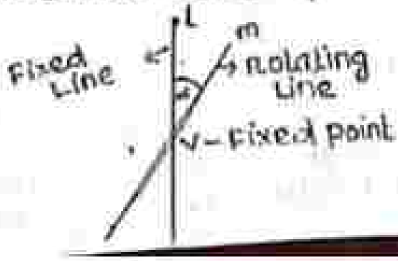
Solⁿ: $2x^2-(k-1)y^2-6x+4y-1=0$ —— ①

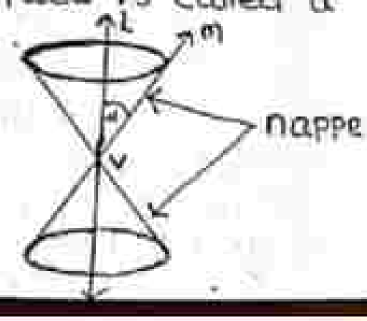
Eqⁿ(1) will represent a circle if

$$2=-(k-1) \Rightarrow 2=-k+1 \Rightarrow k=1-2=-1 \text{ (ANS)}$$

J: CONIC SECTION:

- If a straight line intersects a fixed vertical line at a fixed point and rotates such that the angle between the two lines remains constant, then the resulting surface is called a double right circular cone.

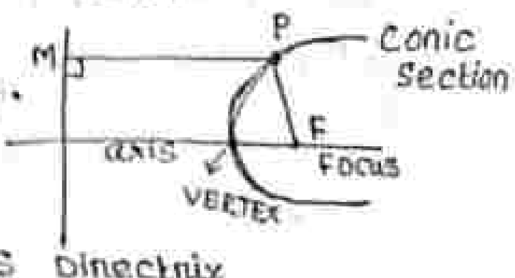




- The fixed vertical line passing through its centre is called axis of the cone (l).
- The rotating line that generates surface of the cone is called generator (m).
- The fixed point at which the generator cuts the axis is called vertex of the cone. (V)
- The cone is divided into two parts by the vertex. The upper part is called upper Nappe and the lower part is called lower Nappe.
- When a plane intersects a cone, it cuts a section from the cone; This section is called conic section or a conic.
- Depending on the position of the intersecting plane and the angle it makes with axis, The conic section can be a circle, an ellipse, a parabola or a hyperbola.
- If the plane intersects one nappe at right angle to the axis then the resulting conic section is a <u>circle</u>.
- If the plane intersects one nappe of the cone at an angle less than 90° but greater than the angle that the generator is inclined with axis, then the resulting conic section is an <u>ellipse</u>.
- If the plane intersects one nappe of the cone such that the plane is parallel to generator then the resulting conic section is a <u>parabola</u>
- If the plane intersects both the nappe of the cone such that the plane is parallel to the axis. then the resulting conic section is a hyperbola.
- 

- The Locus of points which moves in a plane such that its distance from a fixed point always bears a constant ratio to its perpendicular distance from a fixed line in the plane is called conic section.

The fixed point (doesn't lie on fixed line) is called <u>Focus</u>.



The fixed line is called <u>Directrix</u>.

The line passing through focus and perpendicular to directrix is called <u>axis</u>

The point of intersection of conic with axis is called vertex

The constant ratio is called eccentricity denoted by e.

$\therefore \quad \boxed{\frac{PF}{PM} = e}$

conic section	eccentricity
parabola	$e=1$ ($PF=PM$)
Ellipse	$e<1$ ($PF<PM$)
hyperbola	$e>1$ ($PF>PM$)

J₁: GENERAL EQUATION OF CONIC:

- The general equation of conic is

$$\boxed{ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0} \quad \text{——} \; (1)$$

- If $ab-h^2=0 \Rightarrow$ Eqⁿ (1) is a Parabola.
- If $ab-h^2>0 \Rightarrow$ Eqⁿ (1) is an Ellipse.
- If $ab-h^2<0 \Rightarrow$ Eqⁿ (1) is a hyperbola.
- If $a=b, h=0 \Rightarrow$ Eqⁿ (1) is a circle

J₂: PARABOLA:

- Equation of parabola having vertex at origin and Focus at $(a,0)$ is $\boxed{y^2=4ax}$

- Focal chord: A chord of the parabola passing through focus is called focal chord.

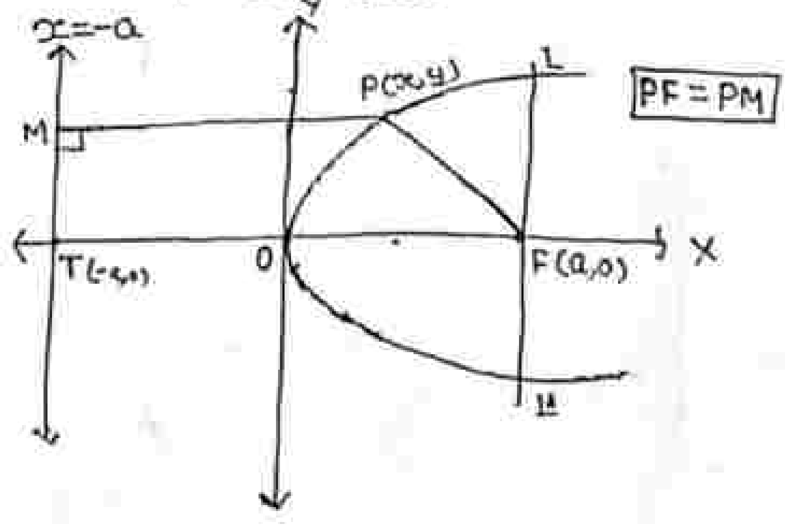
- Focal distance: The distance of a point on the parabola from focus is called focal distance.

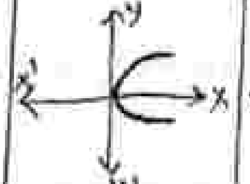
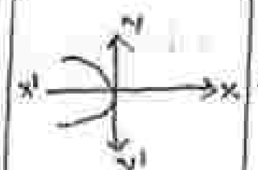
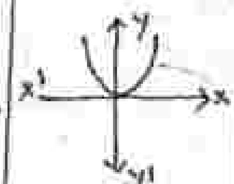
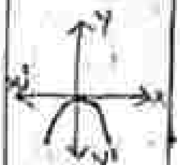
- Latus rectum: The chord passing through focus and perpendicular to axis is called Latus Rectum. (LL')

- Double ordinate: A chord of the parabola perpendicular to axis is called double ordinate.

- Note that vertex is the mid point of focus and the point at which the directrix cuts the axis

$\therefore \quad \boxed{OF = OT}$

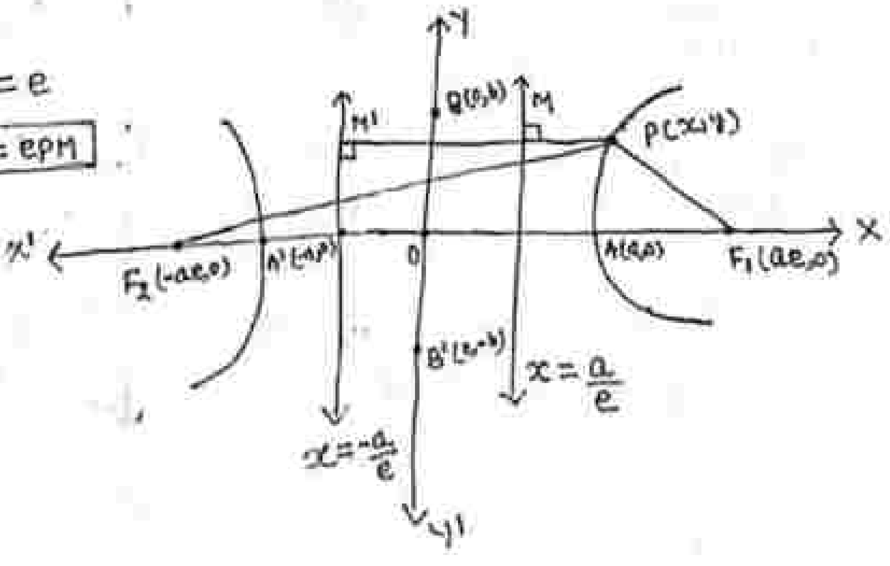


Equation of standard parabolas → / properties ↓	$y^2 = 4ax$ $(a>0)$	$y^2 = -4ax$ $(a>0)$	$x^2 = 4ay$ $(a>0)$	$x^2 = -4ay$ $(a>0)$
vertex	(0,0)	(0,0)	(0,0)	(0,0)
Focus	(a,0)	(-a,0)	(0,a)	(0,-a)
Equation of directrix	$x+a=0$	$x-a=0$	$y+a=0$	$y-a=0$
Length of Latus rectum	4a	4a	4a	4a
AXIS	x-axis	x-axis	y-axis	y-axis
End points of Latus rectum	$(a \pm 2a)$	$(-a \pm 2a)$	$(\pm 2a, a)$	$(\pm 2a, -a)$
Symmetry	about x-axis	about x-axis	about y-axis	about y-axis
graph	 opens to right	 opens to Left	 opens to upward	 opens to downward

J_3 : HYPERBOLA:

- Equation of hyperbola with vertices at $(\pm a, 0)$ and foci at $(\pm ae, 0)$ is $\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$
where eccentricity $e > 1$, $\boxed{b^2 = a^2(e^2 - 1)}$

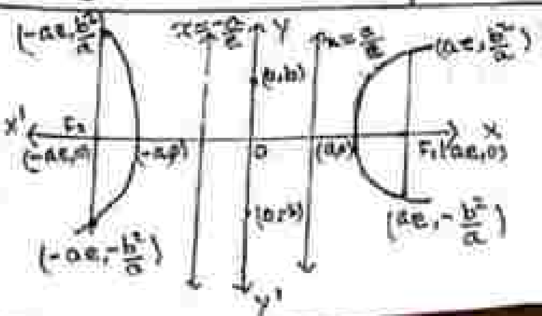
- Here $\frac{PF_1}{PM} = e$
$\Rightarrow \boxed{PF_1 = ePM}$

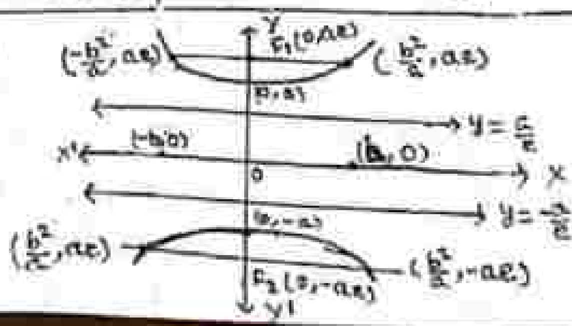


Scanned with CamScanner

- CENTRE: Mid point of two foci is called Centre of the hyperbola. (O)
- Transverse axis The line through two foci and perpendicular to directrix is called Transverse axis or axis of the hyperbola (x-axis).
- Conjugate axis: Line passing through centre and perpendicular to transverse axis is called Conjugate axis (Y-axis).
- Vertices: The points at which the hyperbola intersect the transverse axis are called vertices (A, A')
- If C = length of focci from centre $\boxed{C=\sqrt{a^2+b^2}}$ and $\boxed{e=\frac{c}{a}}$
- The hyperbola doesn't cut conjugate axis at real points but we take two points B(0,b) and B'(0,-b) on conjugate axis Y. These two points are called co-vertices.
- Focal chord, focal distance, Latus rectum, double ordinate are same as parabola.

Equation of standard hypebola → properties↓	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$
Relation between a, b	$b^2=a^2(e^2-1)$	$b^2=a^2(e^2-1)$
Centre	(0,0)	(0,0).
Focci	(ae,0) and (-ae,0)	(0,ae) and (0,-ae)
vertices	(a,0) and (-a,0)	(0,a) & (0,-a)
Transverse axis (Axis)	x axis i.e y=0 Length = 2a	y-axis i.e x=0 Length=2a
Conjugate axis	Y-axis i.e x=0 Length = 2b	X-axis i.e y=0 Length=2b
Length of foci from centre (c)	$c=\sqrt{a^2+b^2}$	$c=\sqrt{a^2+b^2}$
Eccentricity (e)	$e=\frac{c}{a}$	$e=\frac{c}{a}$
End points of Latus rectum	$(ae,\pm\frac{b^2}{a})$ and $(-ae,\pm\frac{b^2}{a})$	$(\pm\frac{b^2}{a},ae)$ and $(\pm\frac{b^2}{a},-ae)$
Length of Latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Equation of directrices	$x=\pm\frac{a}{e}$	$y=\pm\frac{a}{e}$
Symmetry	about both axes	About both axes





- Equilateral or rectangular hyperbola:

If $a=b$ then the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ or $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ is called equilateral hyperbola.

Its eccentricity $\boxed{e=\sqrt{2}}$

- Conjugate hyperbola:

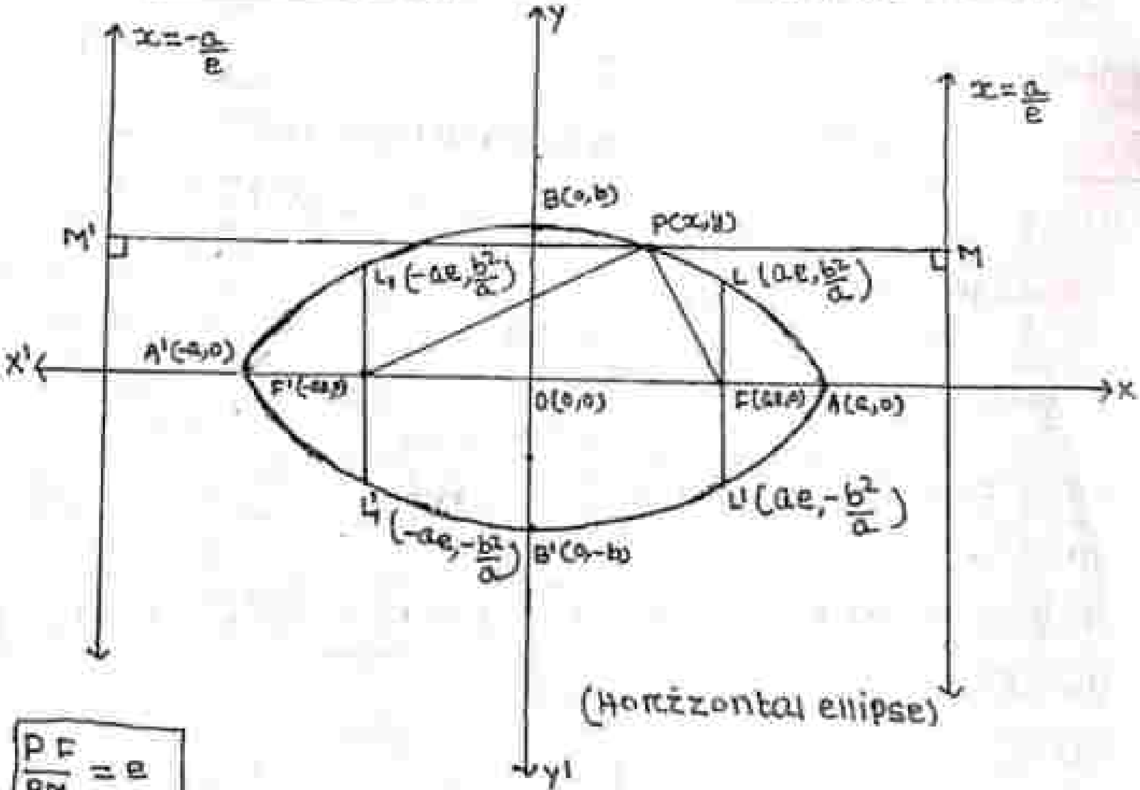
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ is called conjugate hyperbola of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$\frac{y^2}{a^2}-\frac{x^2}{b^2}=-1$ is called conjugate hyperbola of $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$

- Hyperbola is the locus of points in the plane such that difference of its distances from two fixed points (Foci) is constant (2a)
i.e $\boxed{PF_1-PF_2=2a}$

J4: ELLIPSE:

- Equation of Ellipse with vertices at $(\pm a,0)$, Focci at $(\pm ae,0)$ is
$$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$$
where $\boxed{b^2=a^2(1-e^2)}$, $a>b$, eccentricity e is $0<e<1$



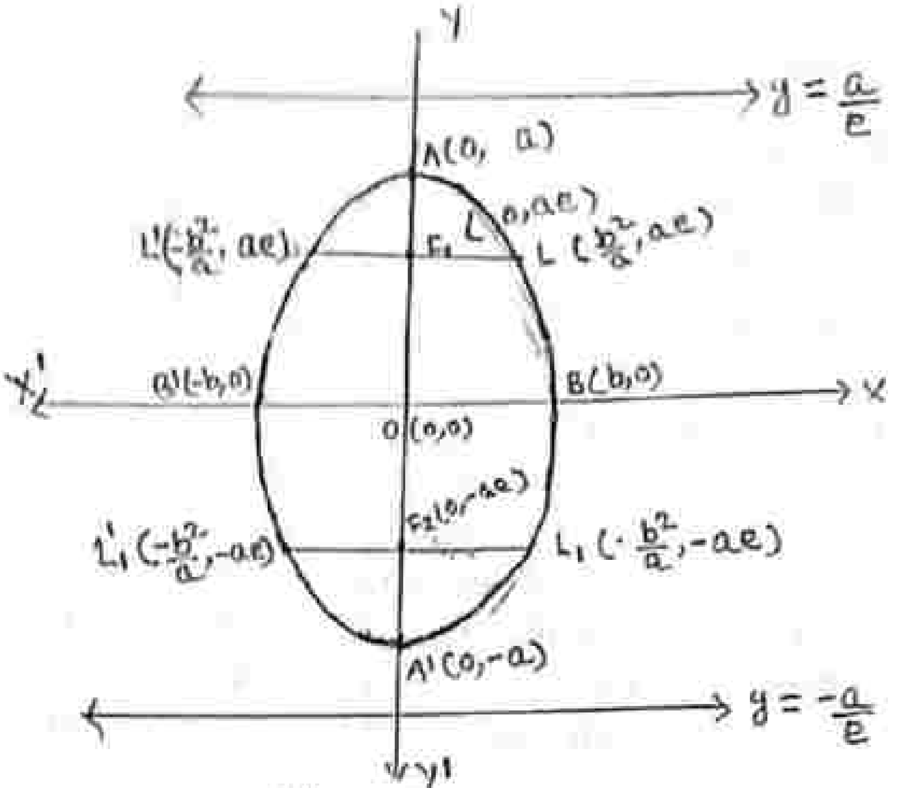
(Horizontal ellipse)

$$\boxed{\frac{PF}{PM}=e}$$

- Major axis: The longest diameter of the ellipse going through centre and two focci from one end to the other, at the broad part of the ellipse is called major axis. ($\overline{AA'}$)
- Minor Axis: The shortest diameter of the ellipse going through centre from one end to other, at the narrow part of the ellipse is called minor axis ($\overline{BB'}$).

- **principal axes:** The major and minor axis together are called principal axes
- **vertices:** The points at which the ellipse cuts major axis are called vertices (A, A')
- **Co-vertices:** The points at which the ellipse cuts minor axis are called co-vertices (B, B')
- centre, focal chord, focal distance, Latus rectum, double ordinate are same as defined in hyperbola.
- **Horizontal ellipse:** If major axis of the ellipse is along x-axis or parallel to x-axis then the ellipse is called Horizontal ellipse.
- **vertical ellipse:** If major axis of the ellipse is along y-axis or parallel to y-axis then the ellipse is called vertical ellipse.
- **oblique Ellipse:** The ellipse whose major axis is neither parallel to x-axis nor parallel to y-axis is called oblique ellipse.
- **semimajor and semiminor axes** · Half of length of major axis and half of length of minor axis are called semi-major and semi-minor axes respectively.

Equation of standard Ellipse (→) properties (↓)	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b$	$\frac{y^2}{a^2}+\frac{x^2}{b^2}=1,\ a>b$
Relation between a & b	$b^2=a^2(1-e^2)$	$b^2=a^2(1-e^2)$
Length of Foci from centre (c)	$c=\sqrt{a^2-b^2}$	$c=\sqrt{a^2-b^2}$
Centre	$(0,0)$	$(0,0)$
Foci	$(\pm ae,0)$	$(0,\pm ae)$
Vertices	$(\pm a,0)$	$(0,\pm a)$
Co-vertices	$(0,\pm b)$	$(\pm b,0)$
eccentricity (e)	$e=\frac{c}{a}$	$e=\frac{c}{a}$
End points of Latus rectum (L.R)	$(ae,\pm\frac{b^2}{a})$ and $(-ae,\pm\frac{b^2}{a})$	$(\pm\frac{b^2}{a},ae)$ and $(\pm\frac{b^2}{a},-ae)$
Length of L.R	$\frac{2b^2}{a}$	$2b^2/a$
Equation of directrices	$x=\pm\frac{a}{e}$	$y=\pm\frac{a}{e}$
Axis	x-axis i.e $y=0$	y-axis i.e $x=0$
Ends of major axis	$(\pm a,0)$	$(0,\pm a)$
Ends of minor axis	$(0,\pm b)$	$(\pm b,0)$
Length of major axis	$2a$	$2a$
Length of minor axis	$2b$	$2b$
Length of semimajor axis	a	a
Length of semiminor axis	b	b
Symmetry	about both axes	about both axes



$\left(\frac{y^2}{a^2}+\frac{x^2}{b^2}=1\right.$, vertical ellipse)

Ex-37: Find co-ordinates of the focus, axis, Equation of directrix and Length of Latus rectum of the parabola $y^2 = 16x$.

Soln: Given parabola is $y^2 = 16x$ —— ①

Comparing this parabola with the parabola $y^2 = 4ax$ we have $4a = 16$

$\Rightarrow a = \frac{16}{4} = 4$.

Focus $= F(a,0) = F(4,0)$. Axis of the parabola is x-axis i.e $y=0$

Equation of directrix is $x = -a \Rightarrow x = -4 \Rightarrow x+4=0$

Length of Latus rectum $= 4a = 4\times4 = 16$ (ANS)

Ex-38: Find Equation of parabola with focus at $(4,0)$ and directrix $x=-4$.

Soln: Focus $= F(4,0) = F(a,0) \Rightarrow a=4$.

Axis is x-axis and parabola opens to right since directrix is $x=-4=-a$

$\therefore$ Eqn of parabola is $y^2 = 4ax \Rightarrow y^2 = 4\times4\times x = 16x$

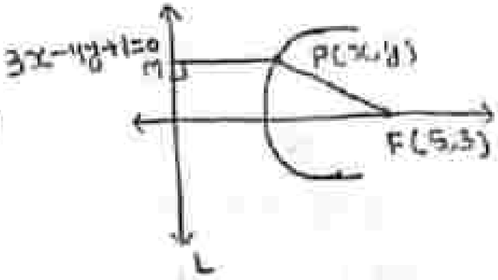
Ex-39: Find Equation of parabola with focus at $(5,3)$ and directrix is $3x-4y+1=0$

Soln: Focus is at $F(5,3)$

Directrix is the Line $L: 3x-4y+1=0$

Let $P(x,y)$ be any point on the parabola.

Draw $\overline{PM} \perp L$ then by definition of parabola



$$\frac{PF}{PM} = e = 1$$

$\Rightarrow$ $PF = PM \Rightarrow \sqrt{(x-5)^2+(y-3)^2} = \left|\frac{3x-4y+1}{\sqrt{(3)^2+(-4)^2}}\right|$

$\Rightarrow$ $25\{(x-5)^2+(y-3)^2\} = (3x-4y+1)^2$

$\Rightarrow$ $25(x^2-10x+25+y^2-6y+9) = 9x^2+16y^2+1-24xy-8y+6x$

$\Rightarrow$ $25x^2-9x^2-250x-6x+850-1+25y^2-16y^2-150y+8y+24xy=0$

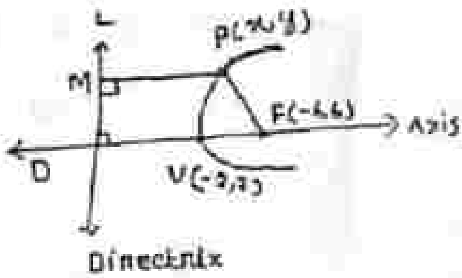
$\Rightarrow$ $16x^2+24xy+9y^2-256x-142y+849=0$ (Ans)

40: Find Equation of parabola whose vertex is $(-2,2)$ and focus is $(-6,6)$

Soln: Vertex of the parabola is at $V(-2,2)$

Focus is at $F(-6,6)$.

Let L is the directrix which cuts the axis $\overleftrightarrow{DF}$ at D. $P(x,y)$ be any point on parabola. $L \perp \overleftrightarrow{DF}$. Draw $\overline{PM} \perp L$.



Let Co-ordinates of D be (a,b)

The Co-ordinates of $V = \left(\frac{a-6}{2}, \frac{b+6}{2}\right)$ $\because$ V is the mid point of $\overline{DF}$

$$\Rightarrow (-2,2) = \left(\frac{a-6}{2}, \frac{b+6}{2}\right)$$

$\Rightarrow$ $\frac{a-6}{2} = -2$ and $\frac{b+6}{2} = 2 \Rightarrow a-6=-4$ and $b+6=4$

$\Rightarrow a=-4+6$ and $b=4-6 \Rightarrow a=2, b=-2$

$\therefore$ Co-ordinate of $D = (2,-2)$

Slope of Line $\overleftrightarrow{DF}$ $(m_1) = \frac{6-(-2)}{-6-2} = \frac{8}{-8} = -1$

$\therefore$ Slope of L $(m) = -\frac{1}{m_1} = -\frac{1}{-1} = 1$ $\because L \perp \overleftrightarrow{DF}$

Equation of L is $y-(-2) = m(x-2)$ $\because$ L passes through $(2,-2)$

$$\Rightarrow y+2 = 1(x-2) = x-2$$

$$\Rightarrow x-y-4=0 \quad \text{——— ①}$$

Now by definition of parabola $PF = PM$

$$\Rightarrow \sqrt{(x+6)^2+(y-6)^2} = \left|\frac{x-y-4}{\sqrt{(1)^2+(-1)^2}}\right|$$

$$2\{(x+6)^2+(y-6)^2\} = (x-y-4)^2$$

$$\Rightarrow 2\{x^2+12x+36+y^2-12y+36\} = x^2+y^2+16-2xy+8y-8x$$

$\Rightarrow 2\{x^2+y^2+12x-12y+72\} = x^2+y^2-8x+8y-2xy+16$

$\Rightarrow 2x^2-x^2+2y^2-y^2+24x+8x-24y-8y+2xy+144-16=0$

$\Rightarrow x^2+y^2+32x-32y+2xy+128=0$ which is the required equation of parabola (ANS)

41: Find co-ordinates of foci, the vertices, eccentricity, length of the Latus rectum of the hyperbola $\frac{x^2}{25}-\frac{y^2}{36}=1$

Soln: Given hyperbola is $\frac{x^2}{25}-\frac{y^2}{36}=1$ —— ①

Comparing ① with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ we have

$$a^2=25 \ \& \ b^2=36 \Rightarrow a=5,\ b=6$$

$\therefore \quad c=\sqrt{a^2+b^2}=\sqrt{5^2+6^2}=\sqrt{61}$

Axis of the hyperbola is x-axis

Foci $=(\pm ae,0)=(\pm c,0)=(\pm\sqrt{61},0)$

vertices $=(\pm a,0)=(\pm 5,0)$

eccentricity $e=\frac{c}{a}=\frac{\sqrt{61}}{5}$

Length of Latus rectum $=\frac{2b^2}{a}=\frac{2\times 6^2}{5}=\frac{72}{5}$ (ANS)

42: Find Equation of hyperbola with foci $(0,\pm 2)$ and vertices $(0,\pm 1)$.

Soln: foci $=F(0,\pm 2)$ and vertices $=V(0,\pm 1)$ are points on y-axis. Hence Axis of the hyperbola is y-axis.

vertices $=V(0,\pm 1)$

$\Rightarrow V(0,\pm a)=V(0,\pm 1)\Rightarrow a=1$

foci $=F(0,\pm 2)\Rightarrow F(0,\pm c)=F(0,\pm 2)\Rightarrow c=2$

But $c^2=a^2+b^2\Rightarrow b=\sqrt{c^2-a^2}=\sqrt{2^2-1^2}=\sqrt{3}$

$\therefore$ Equation of required hyperbola is

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1\Rightarrow\frac{y^2}{1^2}-\frac{x^2}{(\sqrt{3})^2}=1$$

$\Rightarrow y^2-\frac{x^2}{3}=1\Rightarrow 3y^2-x^2=3$ (ANS)

43: Find equation of hyperbola with foci at $(1,2)$, eccentricity $\sqrt{3}$ and directrix $2x+y=1$

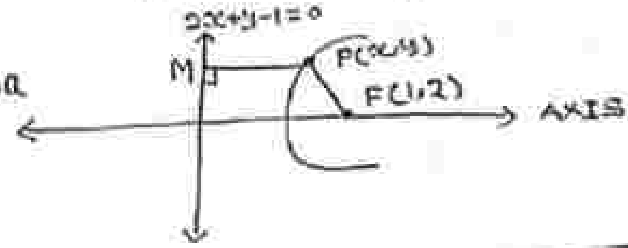
Soln: foci is at $F(1,2)$

eccentricity $(e)=\sqrt{3}$

Directrix is the line $2x+y=1$ i.e $2x+y-1=0$ —— ①

Draw $\overline{PM}\perp$ Line①

By definition of hyperbola



$\frac{PF}{PM}=e$

$\Rightarrow PF = e\,PM \Rightarrow \sqrt{(x-1)^2+(y-2)^2} = \sqrt{3}\cdot\left|\frac{2x+y-1}{\sqrt{2^2+1^2}}\right|$

$\Rightarrow \sqrt{(x-1)^2+(y-2)^2} = \frac{\sqrt{3}}{\sqrt{5}}\cdot|2x+y-1|$

$\Rightarrow (x-1)^2+(y-2)^2 = \frac{3}{5}(2x+y-1)^2$

$\Rightarrow (x^2-2x+1+y^2-4y+4) = \frac{3}{5}(4x^2+y^2+1+4xy-2y-4x)$

$\Rightarrow 5x^2-10x+25+5y^2-20y = 12x^2+3y^2+3+12xy-6y-12x$

$\Rightarrow 5x^2-12x^2-12xy+5y^2-3y^2-10x+12x-20y+6y+25-3=0$

$\Rightarrow -7x^2-12xy+2y^2+2x-14y+22=0$

$\Rightarrow 7x^2+12xy-2y^2-2x+14y-22=0$ (ANS)

11. Find equation of hyperbola having foci at $(0,\pm3)$ and conjugate axis is of length 5.

Soln: Foci are at $F(0,\pm3)$ on y-axis. Thus conjugate axis is x-axis.

$\text{Foci} = F(0,\pm3)$

$\Rightarrow F(0,\pm a) = F(0,\pm3) \Rightarrow a=3$.

Length of conjugate axis $= 5 \Rightarrow 2b=5 \Rightarrow b=\frac{5}{2} \Rightarrow b^2=\frac{25}{4}$

$c^2 = a^2+b^2 \Rightarrow a^2 = c^2-b^2 = 3^2-\left(\frac{5}{2}\right)^2 = 9-\frac{25}{4} = \frac{36-25}{4} = \frac{11}{4}$

Eqn of hyperbola is $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$

$\Rightarrow \frac{y^2}{(11/4)}-\frac{x^2}{25/4}=1 \Rightarrow \frac{4y^2}{11}-\frac{4x^2}{25}=1 \Rightarrow \frac{100y^2-44x^2}{11\times25}=1$

$\Rightarrow 100y^2-44x^2=275$ (ANS)

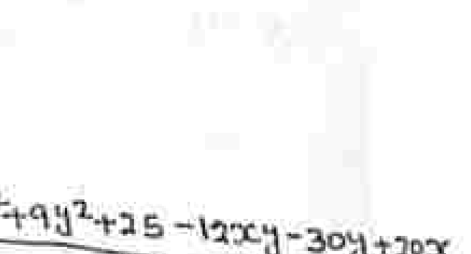
Ex-45: Derive equation of an ellipse whose focus is the point $(-2,2)$, directrix is the line $2x-3y+5=0$ and eccentricity is $\frac{1}{3}$

Soln: Focus is at $F(-2,2)$

Directrix is the line (L): $2x-3y+5=0$ —(i)

eccentricity $e=\frac{1}{3}$

Draw $\overline{PM}\perp L$.



By definition of ellipse $\frac{PF}{PM}=e \Rightarrow PF = e\,PM$

$\Rightarrow \sqrt{(x+2)^2+(y-2)^2} = e\cdot\left|\frac{2x-3y+5}{\sqrt{(2)^2+(-3)^2}}\right|$

$\Rightarrow (x+2)^2+(y-2)^2 = e^2\,\frac{(2x-3y+5)^2}{13}$

$\Rightarrow x^2+4x+4+y^2-4y+4 = \left(\frac{1}{3}\right)^2\cdot\frac{4x^2+9y^2+25-12xy-30y+20x}{13}$

$\Rightarrow 117(x^2+4x+4+y^2-4y+4) = 4x^2+9y^2+25-12xy-30y+20x$

$\Rightarrow 117x^2-4x^2+117y^2-9y^2+468x-20x-468y+30y+12xy+936-25=0$

$\Rightarrow 113x^2+12xy+108y^2+448x-438y+911=0$

46: If Latus rectum of an ellipse is half of its minor axis, find its eccentricity?

Soln: Length of Latus rectum = $\frac{1}{2}$ × minor axis

$\Rightarrow \quad \frac{2b^2}{a} = \frac{1}{2} \times 2b$

$\Rightarrow \quad \frac{2b^2}{a} = b \Rightarrow \frac{2b}{a} = 1 \quad (\because b \neq 0)$

$\Rightarrow \quad a = 2b$

$\Rightarrow \quad a^2 = 4b^2 \Rightarrow a^2 = 4 \cdot a^2(1-e^2) \quad \because b^2 = a^2(1-e^2)$

$\Rightarrow \quad 4(1-e^2) = 1 \Rightarrow 1-e^2 = \frac{1}{4} \Rightarrow e^2 = 1-\frac{1}{4} = \frac{3}{4}$

$\Rightarrow \quad e = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

$\therefore$ eccentricity is $\frac{\sqrt{3}}{2}$ (ANS)

47: Find co-ordinate of foci, vertices, length of major axis, minor axis, eccentricity and Latus rectum of the ellipse

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

Soln: Given ellipse is $\frac{x^2}{9} + \frac{y^2}{4} = 1$ —— ①

Comparing ① with the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ we have

$a^2 = 9$ & $b^2 = 4 \Rightarrow a = 3,\ b = 2$

$c = \sqrt{a^2 - b^2} = \sqrt{3^2 - 2^2} = \sqrt{9-4} = \sqrt{5}$

Foci = $(c, 0)$ and $(-c, 0) = (\sqrt{5}, 0), (-\sqrt{5}, 0)$ since major axis is along x-axis.

Vertices = $(a, 0)$ and $(-a, 0) = (3,0), (-3,0)$

Length of major axis = $2a = 2 \times 3 = 6$ units

Length of minor axis = $2b = 2 \times 2 = 4$ units.

eccentricity (e) = $\frac{c}{a} = \frac{\sqrt{5}}{3}$

Latus rectum = $\frac{2b^2}{a} = \frac{2 \times 2^2}{3} = \frac{8}{3}$

48: Find equation of ellipse whose vertices are $(\pm 10, 0)$ and foci are $(\pm 4, 0)$

Soln: vertices = $V(\pm 10, 0)$ are on x-axis

$\Rightarrow V(\pm a, 0) = V(\pm 10, 0)$

$\Rightarrow \quad a = 10$

Foci = $F(\pm 4, 0) \Rightarrow F(\pm c, 0) = F(\pm 4, 0) \Rightarrow c = 4$

$b = \sqrt{a^2 - c^2} = \sqrt{10^2 - 4^2} = \sqrt{84}$

Eqn of Ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{100} + \frac{y^2}{84} = 1$ (ANS)

EXERCISE :

(1) Find circum centre of the triangle having vertices $(4,3)$, $(-2,3)$, $(6,-1)$.

(2) Find distance of the $(2,5)$ from the line $3x+y+4=0$ measured parallel to a line having slope $3/4$

(3). Find the equation of the line passing through the point of intersection of $x+2y=5$ and $x-3y=7$ and passes through the point $(2,-3)$

(4) Find distance of the point $(-1,-2)$ from the line $x+3y-7=0$ measured parallel to the line $3x+2y-5=0$

(5) Find the perpendicular distance between the lines $3x+4y+5=0$ and $6x+8y+34=0$

(6) Find length of the perpendicular drawn from the point $(1,-2)$ to the line $3x-4y-2=0$

(7) Find equation of lines parallel to the line $3x+4y+1=0$ and at a distance 2 units from the point $(-1,2)$

(8) Find equation of circle passing through the points $(1,0)$, $(0,1)$ and $(0,0)$

(9) Find equation of circle passing through the points $(0,2)$, $(3,0)$ and $(3,2)$

(10) If equation of two diameter of a circle $x-y=5$ and $2x+y=4$ and radius of the circle is 5 then find equation of circle.

(11) Find equation of circle whose centre lies on positive direction of y-axis at a distance 5 from origin and whose radius is 3

(12) Find centre and radius of the circle $25x^2+25y^2-30x-10y-6=0$

(13) prove that the points $(2,-4)$, $(3,-1)$, $(3,-3)$ and $(0,0)$ are concyclic (Hints: They lie on a circle)

(14) Find equation of circle whose centre is at $(2,3)$ and radius 4.

(15) Find equation of circle with centre at $(1,4)$ and passing through a point $(2,6)$.

16. Find equation of circle whose centre is at $(5,5)$ and touches both axes.

17. Determine which circle is greater.
$x^2+y^2-3x+4y=0$ and $x^2+y^2-6x+8y=0$

18. Find equation of circle concentric with the circle $2x^2+2y^2+8x+12y-25=0$ and having its circumference 6π units.

19. Find equation of circle concentric with the circle $4x^2+4y^2-24x+16y-9=0$ and whose area is 9π square units.

20. Find equation of circle passing through the points $(3,-2)$, $(-2,0)$ and whose centre lies on the line $2x-y=3$

21. Find equation of circle circumscribing the triangle having vertices $(1,-5)$, $(5,7)$, $(-5,1)$

22. Find equation of circle whose ends of a diameter are the points $(1,2)$ and $(-3,4)$.

23. Find equation of circle passing through origin and making intercepts 4 and 5 on coordinate axes.

24. For what values of k the equation $x^2+(k-1)xy+y^2+5x-3y+4=0$ will represents a circle.

25. Find equation of parabola having
(i) vertex at $(0,0)$ and focus at $(0,-3)$
(ii) Focus at $(-3,0)$ and directrix $x+5=0$
(iii) vertex at $(4,1)$ and focus at $(4,-3)$
(iv) Focus at $(0,0)$ and directrix is $2x+y-1=0$

26. Find focus, directrix, equation of axis, length of latus rectum, end points of latus rectum of following parabolas
(i) $y^2=16x$, (ii) $y^2=-4x$ (iii) $x^2=4y$ (iv) $4x^2+y=0$

26. Find equation of ellipse having

(i) Foci at $(\pm 4,0)$ and vertices at $(\pm 5,0)$

(ii) major axis is of length 26 and foci at $(\pm 5,0)$

(iii) Centre at $(0,0)$ foci at $(2,0)$ and directrix is $2x-7=0$

(iv) vertices at $(\pm 5,0)$ and Length of Latus rectum is $\frac{8}{5}$.

27. Find vertices, foci, end points of Latus rectum, Length of Latus rectum, equation of axis, centre, directrices, eccentricity of following ellipse.

(i) $\frac{x^2}{36}+\frac{y^2}{16}=1$ (ii) $\frac{x^2}{4}+\frac{y^2}{25}=1$

(28) Find equation of hyperbola having

(i) vertices at $(\pm 5,0)$, foci at $(\pm 7,0)$

(ii) vertices at $(0,\pm 7)$ and eccentricity $=\frac{4}{3}$

(iii) foci at $(0,\pm\sqrt{10})$ and hyperbola passes through $(2,3)$

(29) Find foci, vertices, eccentricity, Equation of directrices and axis, centre, Lenght and end points of Latus rectum, of following hyperbolas

(i) $16x^2-9y^2=144$ (ii) $y^2-3x^2=4$ (iii) $\frac{x^2}{4}-\frac{y^2}{9}=1$

(30) Find Angle between the Lines $3x-y+4=0$ and $2x+y-1=0$

UNIT-IV: VECTOR ALGEBRA:

A: INTRODUCTION: In our real life situation we deal with physical quantities such as mass, length, time, speed, velocity, force, acceleration etc. We can classify these physical quantities into two types i.e scalar quantity and vector quantity.

B: DIRECTED LINE SEGMENT:

→ A directed line segment is a line segment with an arrow head showing direction. Two end points are called initial and terminal points.

→ A directed line segment whose initial point is 'A' and terminal point is 'B' is $\overrightarrow{AB}$

→ A directed line segment has three character i.e Length, sense and support.

→ Length of $\overrightarrow{AB}$ = Length of $\overline{AB}$ = $|\overrightarrow{AB}|$

→ Sense of $\overrightarrow{AB}$ is from A to B

→ Support of $\overrightarrow{AB}$ is the line 'p' of indefinite length of which AB is a portion.

→ Note that $\overrightarrow{AB}$ and $\overrightarrow{BA}$ have same length, support but have opposite sense. Hence $\overrightarrow{AB} \neq \overrightarrow{BA}$

C: SCALAR QUANTITY:

→ A physical quantity which has only magnitude is called scalar quantity.

→ Mass, length, speed, Kinetic energy, temperature are scalar quantities.

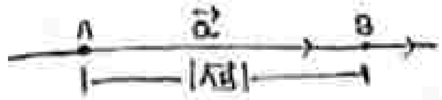
D: VECTOR QUANTITY:

→ A physical quantity which has both magnitude and direction is called vector quantity.

→ Weight, force, velocity, displacement are vector quantity.

E: REPRESENTATION OF A VECTOR:

→ A vector can be represented by a directed line segment. Thus $\overrightarrow{AB}$ represents a vector whose magnitude is AB and direction is from A to B.



→ Notation:

A vector is denoted by a single small letter with an arrow ($\rightarrow$) over it such as $\vec{a}$ or $\vec{a}$ or bold letter **a**

F: TYPES OF VECTOR:

F_1: Null vector:

→ A vector $\vec{a}$ is said to be null vector if $\boxed{|\vec{a}| = 0}$

→ A null vector has arbitrary direction

F_2: UNIT VECTOR:

→ A vector $\vec{a}$ is said to be unit vector if $\boxed{|\vec{a}| = 1}$

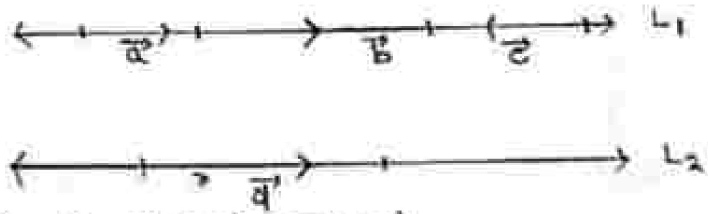
→ If $\hat{a}$ (a cap) is the unit vector along the direction of vector $\vec{a}$ then

$$\boxed{\hat{a} = \frac{\vec{a}}{|\vec{a}|}}$$

F_3: PARALLEL VECTORS:

→ Two vectors are said to be parallel vectors if they act in same line or two parallel lines in same direction. If they act in opposite direction then they are called anti parallel

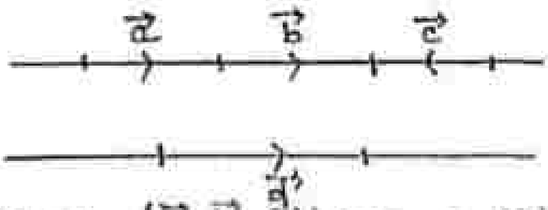
→ $L_1 \parallel L_2$.



Here $\vec{a} \parallel \vec{b}$, $\vec{a} \parallel \vec{d}$,
$\vec{b} \parallel \vec{d}$ but vector $\vec{a} \nparallel \vec{c}$, $\vec{b} \nparallel \vec{c}$, $\vec{c} \nparallel \vec{d}$.

F_4: COLINEAR VECTOR:

→ If two vectors act along same line or along parallel lines then they are called Co-Linear vector



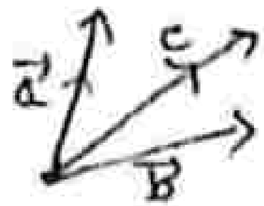
In figure $(\vec{a}, \vec{b}, \vec{c})$ ~~are~~ or $(\vec{a}, \vec{c})$, $(\vec{b}, \vec{c})$ $(\vec{a}, \vec{d})$, $(\vec{c}, \vec{d})$, $(\vec{b}, \vec{d})$, ~~and~~ $(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ are co-linear vectors.

→ Co-linear vectors may be parallel or antiparallel.

Scanned with CamScanner

F5: CO-INITIAL VECTOR:

Two or more vectors having same initial points are called co-initial vectors

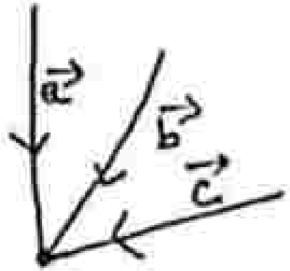


F6: FREE VECTOR

If the value of a vector depends only on its length and direction and is independent of its position in the space, it is called free vectors

F7: COTERMINOUS VECTOR:

vectors having same terminal points are called coterminous vectors



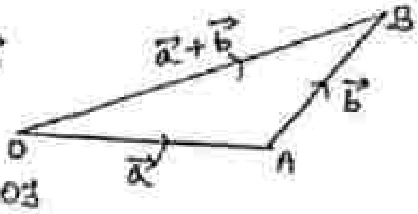
G: VECTOR ADDITION AND SUBSTRACTION:

There are three laws associated with vector addition:

G₁: TRIANGLE LAW OF VECTOR ADDITION:

→ If two vectors are represented by two sides of a triangle taken in same order then their sum (Resultant) is represented by the third side of the triangle taken in opposite order.

→ In triangle OAB vector $\vec{a}$ and $\vec{b}$ are represented by two sides $\vec{OA}$ and $\vec{OB}$ taken in same order. Hence by triangle law of vector addition, $\vec{a}+\vec{b}$ must be represented by 3rd side $\vec{OB}$ which is in opposite order.

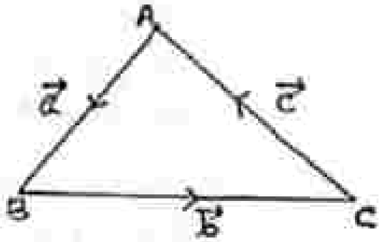


$\vec{OA}=\vec{a}$, $\vec{OB}=\vec{b}$ then $\vec{a}+\vec{b}=\vec{OB}$

Note that $|\vec{OB}| \neq |\vec{OA}|+|\vec{AB}|$

→ Sum of three vectors represented by three sides of a triangle taken same order must be zero vector($\vec{0}$).

i.e $\boxed{\vec{a}+\vec{b}+\vec{c}=\vec{0}}$



vector $\vec{a}=\vec{AB}$, $\vec{CA}=\vec{c}$

$\vec{b}=\vec{BC}$

$\Rightarrow \vec{a}+\vec{b}=\vec{AC}$

$\Rightarrow \vec{a}+\vec{b}=-\vec{CA}$

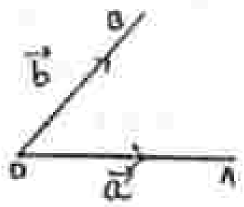
$\Rightarrow \vec{a}+\vec{b}=-\vec{c} \Rightarrow \vec{a}+\vec{b}+\vec{c}=\vec{0}$

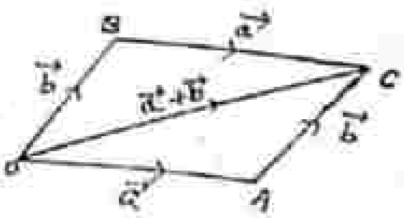
G₂: PARALLELOGRAM LAW OF VECTOR ADDITION:

→ If two vectors are represented by two adjacent sides of a parallelogram in magnitude and direction, then their sum is represented in magnitude and direction by the diagonal of the parallelogram through their common initial point.

→ Let two vectors acting at 'O' are represented by $\vec{OA}$ and $\vec{OB}$. Draw $\vec{AC} \parallel \vec{OB}$ and $\vec{BC} \parallel \vec{OA}$. Then OACB is a parallelogram. Hence $\vec{OA}=\vec{BC}=\vec{a}$ and $\vec{OB}=\vec{AC}=\vec{b}$. Then by ~~parallel~~ Triangle law of vector addition.

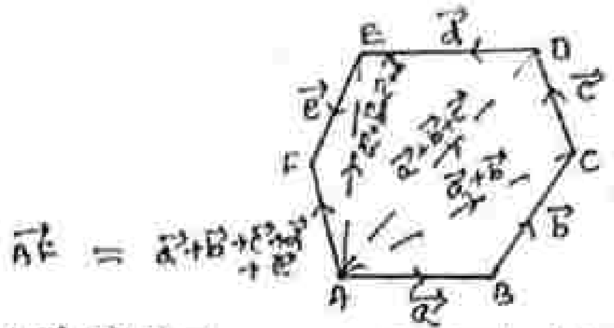
$$\vec{OC}=\vec{OA}+\vec{OB}=\vec{a}+\vec{b}$$





G3: POLYGON LAW OF VECTOR ADDITION:

→ If a number of vectors are represented in magnitude and direction by sides of a polygon taken in order then their sum is represented by closing sides of the polygon taken in opposite order.



→ vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{e}$ are represented by sides $\vec{AB}, \vec{BC}, \vec{CD}, \vec{DE}$ and $\vec{EF}$ respectively of a polygon.

Now by triangle law of vector addition,

In triangle ABC, $\vec{AC} = \vec{AB}+\vec{BC} = \vec{a}+\vec{b}$

In triangle ADC, $\vec{AD} = \vec{AC}+\vec{CD} = \vec{a}+\vec{b}+\vec{c}$

In triangle ADE, $\vec{AE} = \vec{AD}+\vec{DE} = \vec{a}+\vec{b}+\vec{c}+\vec{d}$

In triangle AEF, $\vec{AF} = \vec{AE}+\vec{EF} = \vec{a}+\vec{b}+\vec{c}+\vec{d}+\vec{e}$

→ Note that sum of vectors determined by sides of any polygon taken in order is $\vec{0}$.

In above figure let $\vec{FA} = \vec{f}$.

then $\vec{AF} = \vec{AB}+\vec{BC}+\vec{CD}+\vec{DE}+\vec{EF}$

$\Rightarrow -\vec{FA} = \vec{a}+\vec{b}+\vec{c}+\vec{d}+\vec{e}$

$\Rightarrow -\vec{f} = \vec{a}+\vec{b}+\vec{c}+\vec{d}+\vec{e}$

$\Rightarrow \vec{a}+\vec{b}+\vec{c}+\vec{d}+\vec{e}+\vec{f} = \vec{0}$

or $\vec{AB}+\vec{BC}+\vec{CD}+\vec{DE}+\vec{EF}+\vec{FA} = \vec{0}$.

G4: PROPERTIES:

(i) $\vec{a}+\vec{b} = \vec{b}+\vec{a}$ (Commutative property)

(ii) $(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$ (Associative property)

(iii) $\vec{a}+\vec{0} = \vec{0}+\vec{a} = \vec{a}$ (identity property)

$\vec{0}$ is called additive identity element.

(iv) $\vec{a}+(-\vec{a}) = \vec{0}$ (Additive inverse property)

$-\vec{a}$ is called additive inverse of $\vec{a}$.

(v) $\vec{a}-\vec{b} = \vec{a}+(-\vec{b})$

H: SCALAR MULTIPLICATION WITH A VECTOR:

→ If $\vec{a}$ is any vector and m is a scalar then $m\vec{a}$ (or $\vec{a}m$) is a vector whose (i) $|m\vec{a}| = m|\vec{a}|$ if $m \geq 0$
$= -m|\vec{a}|$ if $m < 0$

(ii) support of $m\vec{a}$ is same as support of $\vec{a}$ or parallel to support of $\vec{a}$.

(iii) Sense of vector $m\vec{a}$ = Sense of $\vec{a}$ if $m>0$
= opposite to sense of $\vec{a}$ if $m<0$

→ Note that $m(n\vec{a}) = (mn)\vec{a}$.
$$(m+n)\vec{a} = m\vec{a} + n\vec{a}$$
$$m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$$

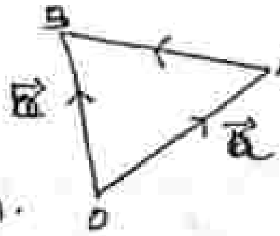
I: POSITION VECTOR: (PV)

→ Let 'O' be the origin (point of Reference) and P be any point then,

$\boxed{\text{position vector of point } P = \vec{OP}}$

→ If position vectors of points A and B are $\vec{a}$ and $\vec{b}$ respectively then $\boxed{\vec{AB} = \vec{b} - \vec{a} = \text{PV of } B - \text{PV of } A}$

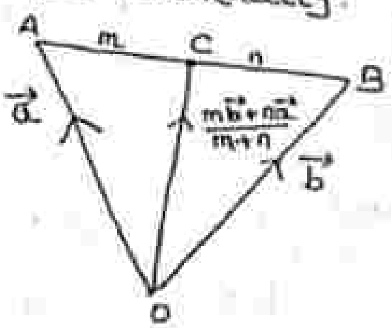
By triangle Law of vector addition,
$$\vec{OB} = \vec{OA} + \vec{AB}$$
$$\Rightarrow \vec{b} = \vec{a} + \vec{AB}$$
$$\Rightarrow \vec{AB} = \vec{b} - \vec{a} = \text{PV of } B - \text{PV of } A.$$

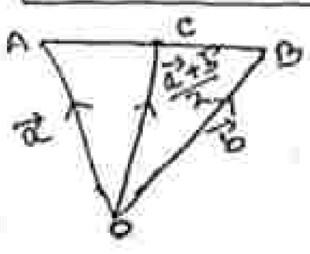


J: SECTION FORMULA: MID-POINT FORMULA

→ If $\vec{a}$ and $\vec{b}$ are position vectors of two points A and B then position vector of point 'C' which divides AB in the ratio $m:n$ is
$$\vec{c} = \boxed{\frac{m\vec{b}+n\vec{a}}{m+n}}$$
If $m:n$ is +ve ⇒ 'C' divides internally.
If $m:n$ is −ve ⇒ C divides externally.

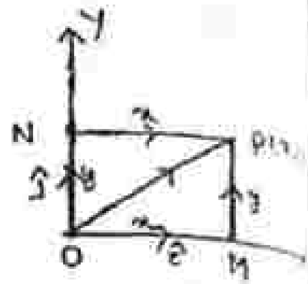
→ If P.V of $A = \vec{a}$ and pv of $B = \vec{b}$
'C' is the mid point of $\overline{AB}$
then $\boxed{\text{pv of } c = \frac{\vec{a}+\vec{b}}{2}}$





K: COMPONENTS OF A VECTOR:

K_1: In two dimension:



→ Let $P(x,y)$ be any point in xoy plane. Draw $\overline{PM} \perp \overline{OX}$, $\overline{PN} \perp \overline{OY}$. Join O and P.
Then $OM = PN = x$, $PM = ON = y$.

→ Let $\hat{i}$ and $\hat{j}$ are unit vectors along x-axis and y-axis respectively.

Then
$$\hat{i} = \frac{\overrightarrow{OM}}{|\overrightarrow{OM}|} = \frac{\overrightarrow{OM}}{x} \Rightarrow \overrightarrow{OM} = x\hat{i}.$$
$$\hat{j} = \frac{\overrightarrow{ON}}{|\overrightarrow{ON}|} = \frac{\overrightarrow{ON}}{y} \Rightarrow \overrightarrow{ON} = y\hat{j}$$

→ Now by triangle law of vector addition
$$\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP}$$
$$\Rightarrow \overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{ON}$$
$$\Rightarrow \boxed{\overrightarrow{OP} = x\hat{i} + y\hat{j}}$$

→ Now in the right angle triangle OMP,
$$OP^2 = OM^2 + PM^2$$
$$\Rightarrow OP = \sqrt{OM^2 + PM^2} = \sqrt{x^2+y^2}$$
$$\Rightarrow \boxed{|\overrightarrow{OP}| = \sqrt{x^2+y^2}}$$

→ If $P(x,y)$ is any point then (a) position vector of $P = \overrightarrow{OP} = x\hat{i} + y\hat{j}$

(b) $|\overrightarrow{OP}| = \sqrt{x^2+y^2}$

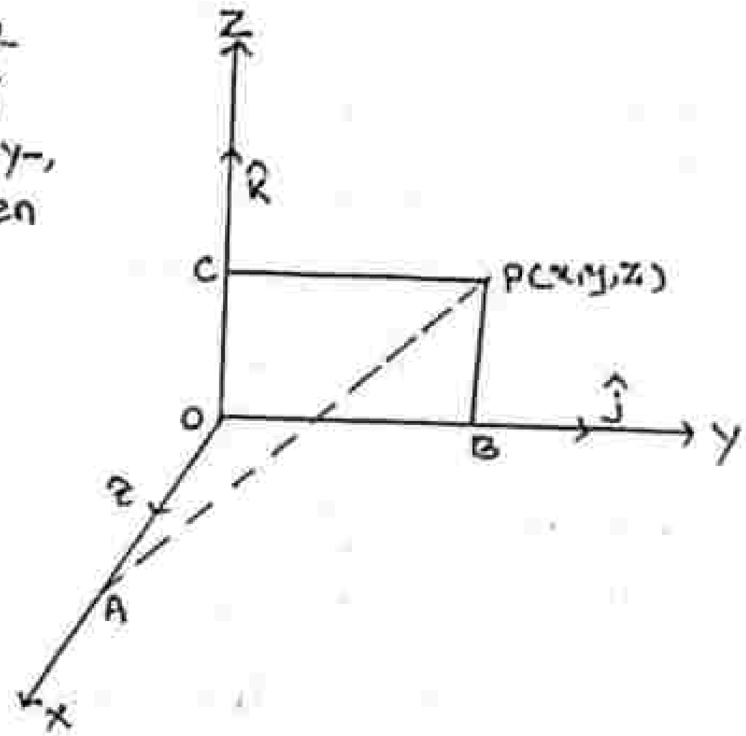
(c) $x\hat{i}$ = vector component of $\overrightarrow{OP}$ along x-axis whose magnitude is x and direction is along $\overrightarrow{OX}$ if $x>0$ and along $\overrightarrow{OX'}$ if $x<0$. Similarly $y\hat{j}$ = vector component of $\overrightarrow{OP}$ along y-axis whose magnitude is y and direction is along $\overrightarrow{OY}$ if $y>0$ and along $\overrightarrow{OY'}$ if $y<0$.

(d) If $\vec{a}$ is a vector then we can write it as
$$\boxed{\vec{a} = a_1\hat{i} + a_2\hat{j} = (a_1, a_2)}$$
$$\boxed{|\vec{a}| = \sqrt{a_1^2 + a_2^2}}$$

(e) If $A(x_1,y_1)$ and $B(x_2,y_2)$ are any two points in the plane then
$$\boxed{\overrightarrow{AB} = (x_2-x_1)\hat{i} + (y_2-y_1)\hat{j}}$$
$$\boxed{|\overrightarrow{AB}| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}}$$

K_2: In three dimension:

→ Similarly if $P(x, y, z)$ is a point in space and $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along x-, y-, z- axes respectively then position vector of P is



$$\boxed{\vec{OP} = x\hat{i} + y\hat{j} + z\hat{k}}$$

$$\boxed{|\vec{OP}| = \sqrt{x^2+y^2+z^2}}$$

$x\hat{i}, y\hat{j}, z\hat{k}$ are vector components of $\vec{OP}$ along co-ordinate axes

→ If $\vec{a}$ is a vector then $\boxed{\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}} = (a_1, a_2, a_3)$

& $\boxed{|\vec{a}| = \sqrt{(a_1)^2+(a_2)^2+(a_3)^2}}$

→ If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are any two points then

$$\boxed{\begin{aligned}\vec{AB} &= (x_2-x_1)\hat{i} + (y_2-y_1)\hat{j} + (z_2-z_1)\hat{k} \\ |\vec{AB}| &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}\end{aligned}}$$

K3: MAGNITUDE AND DIRECTION:

- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ is a vector then magnitude of $\vec{a}$ is

$$\boxed{|\vec{a}| = \sqrt{a_1^2+a_2^2+a_3^2}}$$

- Direction of $\vec{a} = \langle b_1, b_2, b_3 \rangle$ where $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is the unit vector along the direction of $\vec{a}$.

- If $\vec{a} = a_1\hat{i} + a_2\hat{j}$ them

$$\boxed{|\vec{a}| = \sqrt{a_1^2+a_2^2}}$$

Direction $\theta = \tan^{-1}\left(\frac{a_2}{a_1}\right)$ if $(a_1, a_2) \in Q_1$

$= \pi - \tan^{-1}\left(\frac{a_2}{a_1}\right)$, $(a_1, a_2) \in Q_2$

$= \frac{3\pi}{2} - \tan^{-1}\left(\frac{a_2}{a_1}\right)$, $(a_1, a_2) \in Q_3$

$= 2\pi - \tan^{-1}\left(\frac{a_2}{a_1}\right)$, $(a_1\ a_2) \in Q_4$

- Direction of a vector is the angle made by the vector with positive x-axis measured anticlockwisely.

L: SCALAR PRODUCT (DOT PRODUCT)

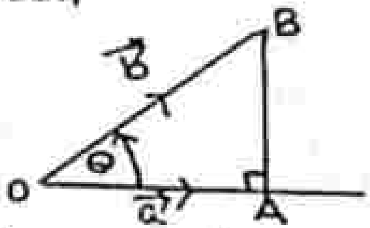
⇒ If $\vec{a}$ and $\vec{b}$ are two vectors and 'θ' is the angle between them then dot product of $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}\cdot\vec{b}$ and defined by

$$\boxed{\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta}$$

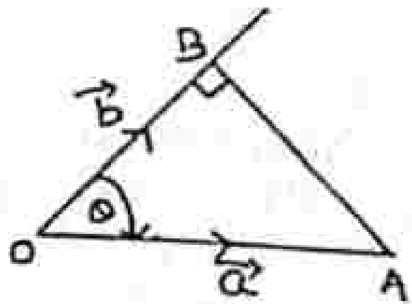
Then
$$\boxed{\theta = \cos^{-1}\left[\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}\right]}$$

⇒ In fig (i) $\overline{BA} \perp \overrightarrow{OA}$

OA is called scalar projection of $\vec{b}$ on $\vec{a}$. $OA\cdot\hat{a}$ is called vector projection of $\vec{b}$ on $\vec{a}$.
$\hat{a}$ = unit vector along $\vec{a}$



(i)



(ii)

In fig (ii) $\overline{AB} \perp \overline{OB}$

OB is called scalar projection of $\vec{a}$ on $\vec{b}$. $OB\cdot\hat{b}$ = vector projection of $\vec{a}$ on $\vec{b}$, $\hat{b}$ = unit vector along $\vec{b}$

In fig (i) $\cos\theta = \frac{OA}{OB} \Rightarrow OA = OB\cdot\cos\theta = |\vec{b}| \times \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

Similarly in fig (ii) $OB = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

⇒ Hence

$$\text{scalar projection of } \vec{a} \text{ on } \vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$$

$$\text{scalar projection of } \vec{b} \text{ on } \vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

AND

$$\text{vector projection of } \vec{a} \text{ on } \vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\cdot\hat{b}$$

$$\text{vector projection of } \vec{b} \text{ on } \vec{a} = \frac{\vec{a}\cdot\vec{b}}{\vec{a}}\cdot\hat{a}$$

⇒ Note that if $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then

$$\boxed{\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3}$$

⇒ $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

⇒ If θ = Angle between $\vec{a}$ and $\vec{b}$ then

$$\boxed{\theta = \cos^{-1}\left[\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}\right]}$$

⇒ $\boxed{\vec{a} \perp \vec{b} \Leftrightarrow a_1b_1 + a_2b_2 + a_3b_3 = 0 \text{ or } \vec{a}\cdot\vec{b} = 0}$

$\boxed{\vec{a} \parallel \vec{b} \Leftrightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} \text{ or } \vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|}$

⇒ $\vec{a}\cdot\vec{a} = |\vec{a}|^2$

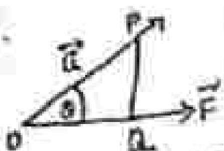
$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$

L.: APPLICATION OF DOT PRODUCT (WORK DONE):

- A force acting on a particle is said to do work, if the particle is displaced in a direction which is not perpendicular to force. It is a scalar quantity.
- work done = Force × Displacement along Force

work done $\boxed{W = \vec{F}\cdot\vec{d} = |\vec{F}||\vec{d}|\cos\theta}$

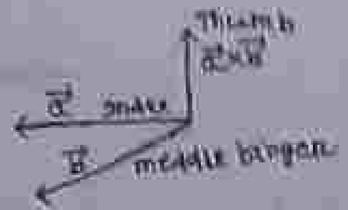


- If $\vec{F}_1, \vec{F}_2 \cdots\cdots \vec{F}_n$ are n forces acting on a particle then during the displacement d of the particle work done by n forces is $\boxed{W = F_1 d + F_2 d + F_3 d \cdots\cdots + F_n d}$. That is these n forces will be replaced by its resultant force $\vec{R}$.

Scanned with CamScanner

M: VECTOR PRODUCT (CROSS PRODUCT)

Right-hand rule: If we can stretch our right hand in such a way that the index, middle finger and thumb are perpendicular to each other then if the first vector ($\vec{a}$) is in direction of index finger, and vector $\vec{b}$ is in direction of middle finger then thumb indicates the direction of unit vector ($\hat{n}$) i.e $\vec{a}\times\vec{b}$



⇒ The vector product of two vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}\times\vec{b}$ and $\vec{a}\times\vec{b}$ is a vector which is perpendicular to both $\vec{a}$ and $\vec{b}$.

⇒ If $\vec{a}$ and $\vec{b}$ are two vectors then

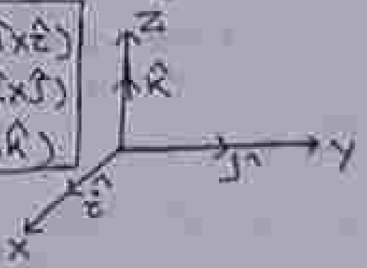
$\boxed{\vec{a}\times\vec{b} = |\vec{a}|\times|\vec{b}|\sin\theta\,\hat{n}}$ where θ = Angle between $\vec{a}$ and $\vec{b}$.

and $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$

⇒ $$\boxed{\hat{n} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}}$$

⇒ Vector product of orthogonal unit vectors ($\hat{i}, \hat{j}, \hat{k}$) forms a right handed system.

$$\boxed{\begin{aligned}\hat{i}\times\hat{j} &= \hat{k} = -(\hat{j}\times\hat{i})\\ \hat{j}\times\hat{k} &= \hat{i} = -(\hat{k}\times\hat{j})\\ \hat{k}\times\hat{i} &= \hat{j} = -(\hat{i}\times\hat{k})\end{aligned}}$$



But $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$

⇒ Note that

a) $\vec{a}\times\vec{b} \neq \vec{b}\times\vec{a}$ (vector product is not commutative)

b) $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$

c) $m(\vec{a}\times\vec{b}) = m\vec{a}\times\vec{b} = \vec{a}\times m\vec{b}$ (m = any scalar)

d) $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$ (Distributive)

(e) $\vec{a}\times\vec{a} = \vec{0}$

(f) If $\vec{a}\parallel\vec{b} \Rightarrow \vec{a}\times\vec{b} = \vec{0}$ ($\because \theta = 0^\circ$)

⇒ $\boxed{\theta = \sin^{-1}\left[\dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}\right]}$

⇒ If $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$
and $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ then

$$\boxed{\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}$$

⇒ Area of triangle with adjacent sides $\vec{a}$ and $\vec{b}$ is

$\boxed{\text{Area} = \frac{1}{2}\times|\vec{a}\times\vec{b}|}$

⇒ Area of parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$ is

$\boxed{\text{Area} = |\vec{a}\times\vec{b}|}$

⇒ Area of parallelogram with diagonal $\vec{a}$ and $\vec{b}$ is

$\boxed{\text{Area} = \frac{1}{2}\times|\vec{a}\times\vec{b}|}$

⇒ Let O be any point and position vector of any point P w.r.t O is $\vec{r}$, $\vec{F}$ = Force. Then moment or torque of the force $\vec{F}$ about O is $\boxed{\vec{M} = \vec{r}\times\vec{F}}$

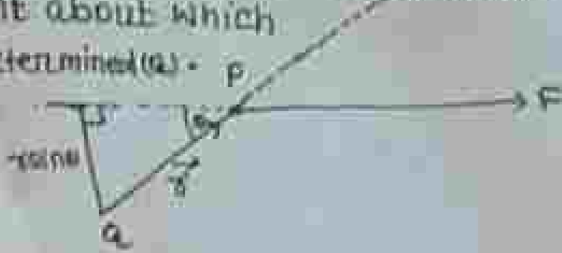
⇒ Three vectors $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, $\vec{c} = c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$ will be co-planar iff

$$\boxed{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0}$$

M.1 APPLICATION OF CROSS PRODUCT: (Moment of Force, Angular velocity)

- **Moment of force:**
  - It is the rotational equivalent of linear force i.e. It is a force causing rotation around a fixed specific point/axis like a door rotating around its hinges. It is also called rotational force or turning effect or torque denoted by τ/M.
  - Moment of force about a point measures the tendency of force to turn the body about that point. If this tendency of rotation is in anti-clockwise direction then moment is positive, otherwise it is (-ve)
  - Let $\vec{F}$ = Force applied to a rigid body at P

    $\vec{r}$ = position vector of the point of application of force (P) w.r.t the point about which moment is to be determined (Q).



θ = Angle between the direction of force and position vector of P w.r.t Q.

Then $\boxed{M = \tau = \vec{r} \times \vec{F}} \Rightarrow M = rF\sin\theta$

$\Rightarrow \boxed{M = r\sin\theta \times F = \text{Force} \times \text{perpendicular distance}}$

$r\sin\theta$ = $\perp^{ar}$ distance of line of action of force from the point where moment is to be calculated (Q) called force arm.

$F\sin\theta$ = component of $\vec{F}$ perpendicular to $\vec{r}$

- It is a vector quantity whose S.I units is Newton-meter (Nm)

ANGULAR VELOCITY:

- The rate of change of angular displacement is called angular velocity denoted by ω.

  $\boxed{\omega = \frac{d\theta}{dt}}$, θ = Angular displacement, t = time.

  Its S.I unit is radian/sec. or revolution/sec. Note that 1 revolution = 2π radians.

- Let a rigid body is rotated about a fixed line OY with an angular velocity $\vec{\omega}$. Let linear velocity of particle M be $\vec{v}$. $\vec{r}$ = position vector of M w.r.t O = $\vec{OM}$

Then $\boxed{\vec{v} = \vec{\omega} \times \vec{r}}$

$\boxed{\vec{\omega} = |\vec{\omega}| \times \text{unit vector along } oy}$

Scanned with CamScanner

PROBLEMS

1. Find the unit vector parallel/along the direction of vector $\hat{i}+3\hat{j}+\hat{k}$.

Ans: Let $\vec{a}=\hat{i}+3\hat{j}+\hat{k} \Rightarrow |\vec{a}| = \sqrt{(1)^2+(3)^2+(1)^2} = \sqrt{11}$

Unit vector along $\vec{a} = \hat{a} = \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{\hat{i}+3\hat{j}+\hat{k}}{\sqrt{11}} = \dfrac{1}{\sqrt{11}}\hat{i}+\dfrac{3}{\sqrt{11}}\hat{j}+\dfrac{1}{\sqrt{11}}\hat{k}$ (Ans)

2. Find unit vector perpendicular to vectors $4\hat{i}-\hat{j}+3\hat{k}$ and $-2\hat{i}+\hat{j}-2\hat{k}$

Ans: Let $\vec{a}=4\hat{i}-\hat{j}+3\hat{k}$

$\vec{b}=-2\hat{i}+\hat{j}-2\hat{k}$

Now $$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix} = \hat{i}(2-3)-\hat{j}(-8+6)+\hat{k}(4-2)$$

$$= -\hat{i}+2\hat{j}+2\hat{k}.$$

$|\vec{a}\times\vec{b}| = \sqrt{(-1)^2+(2)^2+(2)^2} = \sqrt{9} = 3.$

$\therefore$ unit vector perpendicular to $\vec{a}$ and $\vec{b}$

$$= \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} = \frac{-\hat{i}+2\hat{j}+2\hat{k}}{3} = \frac{1}{3}(-\hat{i}+2\hat{j}+2\hat{k}) \text{ (Ans)}$$

3. If $\vec{a}=3\hat{i}+3\hat{j}+\hat{k}$, $\vec{b}=-2\hat{i}+\hat{j}-2\hat{k}$ then find unit vector parallel to $\vec{a}+\vec{b}$.

Ans: $\vec{a}=3\hat{i}+3\hat{j}+\hat{k}$, $\vec{b}=-2\hat{i}+\hat{j}-2\hat{k}$

Now $\vec{a}+\vec{b} = (3\hat{i}+3\hat{j}+\hat{k})+(-2\hat{i}+\hat{j}-2\hat{k})$

$= 3\hat{i}+3\hat{j}+\hat{k}-2\hat{i}+\hat{j}-2\hat{k} = \hat{i}+4\hat{j}-\hat{k}$

$|\vec{a}+\vec{b}| = \sqrt{(1)^2+(4)^2+(-1)^2} = \sqrt{18} = 3\sqrt{2}$

$\therefore$ unit vector parallel to $\vec{a}+\vec{b} = \dfrac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|} = \dfrac{\hat{i}+4\hat{j}-\hat{k}}{3\sqrt{2}}$ (Ans)

4. Find scalar and vector projection of vector $3\hat{i}-\hat{j}-2\hat{k}$ on vector $\hat{i}+2\hat{j}-3\hat{k}$

Ans: Let $\vec{a}=3\hat{i}-\hat{j}-2\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-3\hat{k}$

$\vec{a}\cdot\vec{b} = (3)(1)+(-1)\times(2)+(-2)(-3) = 3-2+6 = 7.$

$|\vec{b}| = \sqrt{(1)^2+(2)^2+(-3)^2} = \sqrt{1+9+4} = \sqrt{14}$

unit vector along $\vec{b}$ is $\hat{b} = \dfrac{\vec{b}}{|\vec{b}|} = \dfrac{\hat{i}+2\hat{j}-3\hat{k}}{\sqrt{14}}$

Now scalar projection of $\vec{a}$ on $\vec{b}$ = $\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{7}{\sqrt{14}} = \frac{7\sqrt{14}}{14}$
$= \frac{\sqrt{14}}{2}$

Vector projection of $\vec{a}$ on $\vec{b}$

$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\cdot\hat{b} = \frac{\sqrt{14}}{2}\times\frac{\hat{i}+2\hat{j}-3\hat{k}}{\sqrt{14}}$

$= \frac{1}{2}(\hat{i}+2\hat{j}-3\hat{k}) = \frac{1}{2}\hat{i}+\hat{j}-\frac{3}{2}\hat{k}$

$\therefore$ scalar projection = $\frac{\sqrt{14}}{2}$
vector projection = $\frac{1}{2}\hat{i}+\hat{j}-\frac{3}{2}\hat{k}$ (Ans)

⑤ Two forces act on a particle at a point. Find the Resultant if they are $4\hat{i}+\hat{j}-3\hat{k}$, and $3\hat{i}+\hat{j}-\hat{k}$

ANS: Let $\vec{F_1} = 4\hat{i}+\hat{j}-3\hat{k}$, $\vec{F_2} = 3\hat{i}+\hat{j}-\hat{k}$

Resultant of $\vec{F_1}$ and $\vec{F_2}$ = $\vec{F_1}+\vec{F_2}$

$= (4\hat{i}+\hat{j}-3\hat{k})+(3\hat{i}+\hat{j}-\hat{k})$

$= 4\hat{i}+\hat{j}-3\hat{k}+3\hat{i}+\hat{j}-\hat{k} = 7\hat{i}+2\hat{j}-4\hat{k}$ (Ans)

⑥ If $|\vec{a}| = 2$, $|\vec{b}| = 5$, $|\vec{a}\times\vec{b}| = 8$ then find $\vec{a}\cdot\vec{b}$.

Soln $|\vec{a}| = 2$, $|\vec{b}| = 5$

Now $|\vec{a}\times\vec{b}| = 8$

$\Rightarrow \left||\vec{a}|\cdot|\vec{b}|\sin\theta\,\hat{n}\right| = 8$ where θ = Angle between $\vec{a}$ & $\vec{b}$, $\hat{n}$ = unit vector $\perp^{r}$ to $\vec{a}$ & $\vec{b}$

$\Rightarrow |2\times5\times\sin\theta\,\hat{n}| = 8$

$\Rightarrow 10\sin\theta\,|\hat{n}| = 8$

$\Rightarrow 10\sin\theta = 8$ $\because |\hat{n}| = 1$

$\Rightarrow \sin\theta = \frac{8}{10} = \frac{4}{5}$

Now $\cos\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}}$

$= \sqrt{\frac{25-16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cdot\cos\theta = 2\times5\times\frac{3}{5} = 6$ (ANS)

(7) Find area of the triangle with vertices $(3,-1,2)$, $(1,-1,3)$, $(4,-3,1)$ by vector method.

Sol:
vertices of the triangle are $A(3,-1,2)$, $B(1,-1,3)$ and $C(4,-3,1)$

now $\vec{a} = \vec{BC} = (4-1)\hat{i} + (-3+1)\hat{j} + (1+3)\hat{k} = 3\hat{i} - 2\hat{j} + 4\hat{k}$

$\vec{b} = \vec{CA} = (3-4)\hat{i} + (-1+3)\hat{j} + (2-1)\hat{k} = -\hat{i} + 2\hat{j} + \hat{k}$

now

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 4 \\ -1 & 2 & 1 \end{vmatrix}$$

$= \hat{i}(-2-8) - \hat{j}(+3+4) + \hat{k}(6-2)$

$= -10\hat{i} - 7\hat{j} + 4\hat{k}$

$|\vec{a} \times \vec{b}| = \sqrt{(-10)^2 + (-7)^2 + (4)^2} = \sqrt{100+49+16} = \sqrt{165}$

Area of the triangle $= \frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2}\sqrt{165}$ sq.unit (Ans)

(8) Find area of the triangle whose two adjacent sides are determined by the vectors $3\hat{i}+4\hat{j}$ and $-5\hat{i}+7\hat{j}$

<u>Solution</u> Adjacent sides of the triangle are

$\vec{a} = 3\hat{i}+4\hat{j} = 3\hat{i}+4\hat{j}+0\hat{k}$

$\vec{b} = -5\hat{i}+7\hat{j} = -5\hat{i}+7\hat{j}+0\hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 0 \\ -5 & 7 & 0 \end{vmatrix} = \hat{i}(0-0) + \hat{j}(0-0) + \hat{k}(21+20) = 41\hat{k}$$

$|\vec{a} \times \vec{b}| = \sqrt{(0)^2+(0)^2+(41)^2} = 41$

Area of the triangle $= \frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2} \times 41 = \frac{41}{2}$ sq.unit.

(9) Find area of the triangle having position vectors of its vertices $\hat{i}-2\hat{j}+3\hat{k}$, $3\hat{i}+\hat{j}+2\hat{k}$, $2\hat{i}+3\hat{j}-\hat{k}$

<u>Solution</u> Let vertices of the triangle be A, B, C and 'O' is the point of reference

Scanned with CamScanner

position vector (p.v) of $A = \vec{OA} = \hat{i} - 2\hat{j} + 3\hat{k}$

p.v of $B = \vec{OB} = 3\hat{i} + \hat{j} + 2\hat{k}$

p.v of $C = \vec{OC} = 2\hat{i} + 3\hat{j} - \hat{k}$

Now $\vec{a} = \vec{BC} = \vec{OC} - \vec{OB} = (2\hat{i} + 3\hat{j} - \hat{k}) - (3\hat{i} + \hat{j} + 2\hat{k})$

$= 2\hat{i} + 3\hat{j} - \hat{k} - 3\hat{i} - \hat{j} - 2\hat{k}$

$= -\hat{i} + 2\hat{j} - 3\hat{k}$

$\vec{b} = \vec{CA} = \vec{OA} - \vec{OC}$

$= (\hat{i} - 2\hat{j} + 3\hat{k}) - (2\hat{i} + 3\hat{j} - \hat{k})$

$= -\hat{i} - 5\hat{j} + 4\hat{k}$

Now $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & -3 \\ -1 & -5 & 4 \end{vmatrix} = \hat{i}(8-15) - \hat{j}(-4-3) + \hat{k}(5+2)$

$= -7\hat{i} + 7\hat{j} + 7\hat{k}$

$|\vec{a} \times \vec{b}| = \sqrt{(-7)^2 + (7)^2 + (7)^2} = \sqrt{3 \times (7)^2} = 7\sqrt{3}$

Area of triangle $= \frac{1}{2} \times |\vec{a} \times \vec{b}| = \frac{1}{2} \times 7\sqrt{3} = \frac{7\sqrt{3}}{2}$ sq. unit (Ans)

(10) Find area of the parallelogram whose adjacent sides are determined by the vectors $\hat{i} + 2\hat{j} + 3\hat{k}$, and $3\hat{i} - 2\hat{j} + \hat{k}$

Soln Adjacent sides of the parallelogram are

$\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$

$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 3 & -2 & 1 \end{vmatrix} = \hat{i}(2+6) - \hat{j}(1-9) + \hat{k}(-2-6)$

$= 8\hat{i} + 8\hat{j} - 8\hat{k} \Rightarrow |\vec{a} \times \vec{b}| = \sqrt{(8)^2 + (8)^2 + (-8)^2} = \sqrt{3(8)^2} = 8\sqrt{3}$

$\therefore$ Area of the parallelogram $= |\vec{a} \times \vec{b}| = 8\sqrt{3}$ sq. units (Ans)

(11) Find area of parallelogram whose diagonals are determined by the vectors $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} - 3\hat{j} + 4\hat{k}$

Soln: Diagonals of the parallelogram are

$\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$, $\vec{b} = \hat{i} - 3\hat{j} + 4\hat{k}$

Now $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = \hat{i}(4-6)-\hat{j}(12+2)+\hat{k}(-9-1) = -2\hat{i}-14\hat{j}-10\hat{k}.$

$|\vec{a}\times\vec{b}| = \sqrt{(-2)^2+(-14)^2+(-10)^2} = \sqrt{4+196+100} = \sqrt{300} = 10\sqrt{3}$

Area of parallelogram $= \frac{1}{2}\times|\vec{a}\times\vec{b}| = \frac{1}{2}\times 10\sqrt{3} = 5\sqrt{3}$ sq.unit (Ans)

(12) Find workdone by the force $\vec{F} = \hat{i}+\hat{j}+\hat{k}$ acting on a particle if the particle is displaced from $A(3,3,3)$ to $B(4,4,4)$

Solⁿ: $\vec{F} = \hat{i}+\hat{j}+\hat{k}.$

The particle is displaced from $A(3,3,3)$ to $B(4,4,4)$.

Hence displacement

$$\vec{s} = \vec{AB} = (4-3)\hat{i}+(4-3)\hat{j}+(4-3)\hat{k} = \hat{i}+\hat{j}+\hat{k}.$$

Workdone $W = \vec{F}\cdot\vec{s} = (1)(1)+(1)(1)+(1)(1) = 3$ unit (Ans).

(13) Show that the points A, B, C with position vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$, and $3\hat{i}-4\hat{j}-4\hat{k}$ respectively are the vertices of a right angle triangle. Also find remai angles of the triangle.

Solⁿ: position vector of $A = \vec{OA} = 2\hat{i}-\hat{j}+\hat{k}$

p.v of $B = \vec{OB} = \hat{i}-3\hat{j}-5\hat{k}$

p.v of $C = \vec{OC} = 3\hat{i}-4\hat{j}-4\hat{k}.$

Now $\vec{AB} = \vec{OB}-\vec{OA} = (\hat{i}-3\hat{j}-5\hat{k})-(2\hat{i}-\hat{j}+\hat{k}) = -\hat{i}-2\hat{j}-6\hat{k}.$

$\vec{BC} = \vec{OC}-\vec{OB} = (3\hat{i}-4\hat{j}-4\hat{k})-(\hat{i}-3\hat{j}-5\hat{k}) = 2\hat{i}-\hat{j}+\hat{k}.$

$\vec{CA} = \vec{OA}-\vec{OC} = (2\hat{i}-\hat{j}+\hat{k})-(3\hat{i}-4\hat{j}-4\hat{k}) = -\hat{i}+3\hat{j}+5\hat{k}$

Now $\vec{AB}+\vec{BC}+\vec{CA} = (-\hat{i}-2\hat{j}-6\hat{k})+(2\hat{i}-\hat{j}+\hat{k})+(-\hat{i}+3\hat{j}+5\hat{k})$

$= -\hat{i}-2\hat{j}-6\hat{k}+2\hat{i}-\hat{j}+\hat{k}-\hat{i}+3\hat{j}+5\hat{k} = \vec{0}$

Hence A, B, C are vertices of a triangle.

Now $\vec{AC} = -\vec{CA} = \hat{i}-3\hat{j}-5\hat{k}$

$\vec{BA} = -\vec{AB} = \hat{i}+2\hat{j}+6\hat{k}.$

$\vec{CB} = -\vec{BC} = -2\hat{i}+\hat{j}-\hat{k}.$

$m\angle A$ = Angle between $\vec{AC}$ and $\vec{AB}$ $= \cos^{-1}\left[\frac{(1)(-1)+(-3)(-2)+(-5)(-6)}{\sqrt{(1)^2+(-3)^2+(-5)^2}\sqrt{(-1)^2+(-2)^2+(-6)^2}}\right]$

$= \cos^{-1}\left[\frac{35}{\sqrt{35}\cdot\sqrt{41}}\right] = \cos^{-1}\left[\sqrt{\frac{35}{41}}\right].$

$\angle B$ = Angle between $\vec{BA}$ and $\vec{BC}$

$$= \cos^{-1}\left[\frac{(1)(2)+(2)(-1)+(6)(1)}{\sqrt{(1)^2+(2)^2+(6)^2}\sqrt{(2)^2+(-1)^2+(1)^2}}\right]$$

$$= \cos^{-1}\left[\frac{6}{\sqrt{41}\sqrt{\sqrt{6}}}\right] = \cos^{-1}\sqrt{\frac{6}{41}}$$

$\angle C$ = Angle between $\vec{CA}$ and $\vec{CB}$

$$= \cos^{-1}\left[\frac{(-1)(-2)+(3)(1)+(5)(-1)}{\sqrt{(-1)^2+(3)^2+(5)^2}\sqrt{(-2)^2+(1)^2+(-1)^2}}\right]$$

$$= \cos^{-1}[0] = 90^\circ$$

since $m\angle C = 90^\circ \Rightarrow$ ABC is a right angle triangle.

Other two angles are $m\angle A = \cos^{-1}\sqrt{\frac{35}{41}}$, $m\angle B = \cos^{-1}\sqrt{\frac{6}{41}}$

(14) Find the angle between the vectors $5\hat{i}+3\hat{j}+4\hat{k}$ and $6\hat{i}-8\hat{j}-\hat{k}$

Sol^n:

Let $\vec{a} = 5\hat{i}+3\hat{j}+4\hat{k}$, $\vec{b} = 6\hat{i}-8\hat{j}-\hat{k}$.

Here $a_1 = 5$, $a_2 = 3$, $a_3 = 4$

$b_1 = 6$, $b_2 = -8$, $b_3 = -1$

Let θ be the angle between $\vec{a}$ & $\vec{b}$ then

$$\theta = \cos^{-1}\left\{\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}\right\}$$

$$= \cos^{-1}\left\{\frac{(5)(6)+(3)(-8)+(4)(-1)}{\sqrt{(5)^2+(3)^2+(4)^2}\sqrt{(6)^2+(-8)^2+(-1)^2}}\right\}$$

$$= \cos^{-1}\left\{\frac{2}{\sqrt{50}\cdot\sqrt{101}}\right\} = \cos^{-1}\left\{\frac{2}{5\sqrt{2}\cdot\sqrt{101}}\right\}$$

$$= \cos^{-1}\left\{\frac{1}{5}\cdot\frac{\sqrt{2}}{\sqrt{101}}\right\} = \cos^{-1}\left\{\frac{\sqrt{202}}{505}\right\} \text{ (Ans).}$$

(15) Find the value of p for which the vectors $\vec{a} = 3\hat{i}+2\hat{j}+9\hat{k}$ and $\vec{b} = \hat{i}+p\hat{j}+3\hat{k}$ are perpendicular.

Sol^n: $\vec{a} = 3\hat{i}+2\hat{j}+9\hat{k}$

$\vec{b} = \hat{i}+p\hat{j}+3\hat{k}$

Here $a_1 = 3$, $a_2 = 2$, $a_3 = 9$

$b_1 = 1$, $b_2 = p$, $b_3 = 3$

$\vec{a} \perp \vec{b} \Rightarrow a_1b_1 + a_2b_2 + a_3b_3 = 0 \Rightarrow (3)(1) + (2)(p) + (9)(3) = 0$

$\Rightarrow 3 + 2p + 27 = 0 \Rightarrow 2p = -30 \Rightarrow p = \frac{-30}{2} = -15$ (Ans)

(18) Find p for which vectors $3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\hat{i} + p\hat{j} + 3\hat{k}$ are parallel.

$\underline{\text{Sol}^n}$: Let $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$

Here
$a_1 = 3$, $a_2 = 2$, $a_3 = 9$
& $b_1 = 1$, $b_2 = p$, $b_3 = 3$

$\vec{a} \parallel \vec{b} \Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} \Rightarrow \frac{3}{1} = \frac{2}{p} = \frac{9}{3}$

$\Rightarrow \frac{2}{p} = 3 \Rightarrow 3p = 2 \Rightarrow p = \frac{2}{3}$ (Ans)

(19) If $\vec{OA} = 6\hat{i} + 3\hat{j} - 2\hat{k}$, $\vec{OB} = 3\hat{i} - 5\hat{j} + \hat{k}$ then find magnitude and direction of $\vec{AB}$

$\underline{\text{Sol}^n}$: $\vec{OA} = 6\hat{i} + 3\hat{j} - 2\hat{k}$, $\vec{OB} = 3\hat{i} - 5\hat{j} + \hat{k}$

$\vec{AB}$ = P.V of B − P.V of A

$= \vec{OB} - \vec{OA} = (3\hat{i} - 5\hat{j} + \hat{k}) - (6\hat{i} + 3\hat{j} - 2\hat{k})$

$= 3\hat{i} - 5\hat{j} + \hat{k} - 6\hat{i} - 3\hat{j} + 2\hat{k} = -3\hat{i} - 8\hat{j} + 3\hat{k}$

Now magnitude of $\vec{AB} = |\vec{AB}| = \sqrt{(-3)^2 + (-8)^2 + (3)^2}$

$= \sqrt{82}$

Again unit vector along $\vec{AB} = \frac{\vec{AB}}{|\vec{AB}|}$

$= \frac{-3\hat{i} - 8\hat{j} + 3\hat{k}}{\sqrt{82}} = \frac{-3}{\sqrt{82}}\hat{i} - \frac{8}{\sqrt{82}}\hat{j} + \frac{3}{\sqrt{82}}\hat{k}$

$\therefore$ Direction of $\vec{AB}$ is $= \left\langle \frac{-3}{\sqrt{82}}, \frac{-8}{\sqrt{82}}, \frac{3}{\sqrt{82}} \right\rangle$ (Ans)

(20) Show that the points A(2, 6, 3), B(1, 2, 7) and C(3, 10, −1) are co-linear.

$\underline{\text{Sol}^n}$: Given points are A(2, 6, 3), B(1, 2, 7), C(3, 10, −1)

Now $\vec{AB} = (1-2)\hat{i} + (2-6)\hat{j} + (7-3)\hat{k}$

$= -\hat{i} - 4\hat{j} + 4\hat{k}$

$\vec{AC} = (3-2)\hat{i} + (10-6)\hat{j} + (-1-3)\hat{k}$

$= \hat{i} + 4\hat{j} - 4\hat{k}$

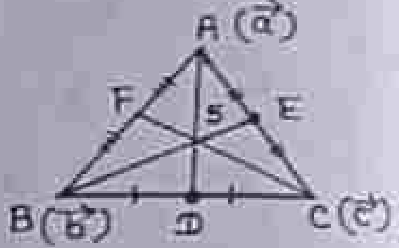
$= -(-\hat{i} - 4\hat{j} + 4\hat{k}) = -\overrightarrow{AB}$

$\therefore \overrightarrow{AC} = (-1)\overrightarrow{AB} \Rightarrow \overrightarrow{AB}$ & $\overrightarrow{AC}$ are parallel or co linear.

But they have a point in common and same support.

Hence A, B, C are co-linear.

(19) Prove by vector method the medians of a Triangle are concurrent.



Soln: Let ABC be the triangle.

D, E, F are mid points of $\overline{BC}, \overline{CA}$, and $\overline{AB}$ respectively.

Let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of A, B, C respectively.

D is the mid point of $\overline{BC}$. Hence

P.V of $D = \dfrac{\vec{b}+\vec{c}}{2}$. similarly

P.V of $E = \dfrac{\vec{a}+\vec{c}}{2}$

P.V of $F = \dfrac{\vec{a}+\vec{b}}{2}$

Let two medians $\overline{AD}$ and $\overline{BE}$ intersects at G.

Hence $AG : GD = 2:1$

$$\therefore \text{P.V of } G = \frac{2 \times \frac{\vec{b}+\vec{c}}{2} + 1 \times \vec{a}}{2+1} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

Now Let H be any point on $\overline{CF}$ which divides $\overline{CF}$ in 2:1 ratio. Then

$$\text{P.V of } H = \frac{2 \times \frac{\vec{a}+\vec{b}}{2} + 1 \times \vec{c}}{2+1} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

$\therefore$ PV of G = PV of H $\Rightarrow G = H$

Hence the median $\overline{CF}$ passes through point of intersections G of $\overline{AD}$ and $\overline{BE}$

$\Rightarrow \overline{AD}, \overline{BE}, \overline{CF}$ are Concurrent.

$\therefore$ medians of a Triangle are concurrent.

(20) If $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$ then show that $\vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ or $\vec{a} \perp (\vec{b}-\vec{c})$

Sol^n: $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c} \Rightarrow \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{c} = \vec{0} \Rightarrow \vec{a}\cdot(\vec{b}-\vec{c}) = \vec{0}$

$\Rightarrow \vec{a} = \vec{0}$ or $\vec{b}-\vec{c} = \vec{0}$ or $\vec{a} \perp (\vec{b}-\vec{c})$

$\Rightarrow \vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ or $\vec{a} \perp (\vec{b}-\vec{c})$ ▭

(21) If $\hat{a}$ and $\hat{b}$ are unit vectors and 'θ' is the angle between them then show that $\sin\left(\frac{\theta}{2}\right) = \frac{1}{2}|\hat{a}-\hat{b}|$.

Sol^n: $|\hat{a}-\hat{b}|^2 = (\hat{a}-\hat{b})\cdot(\hat{a}-\hat{b})$ $\quad \because |\vec{a}|^2 = \vec{a}\cdot\vec{a}$

$= \hat{a}\cdot\hat{a} - \hat{a}\cdot\hat{b} - \hat{b}\cdot\hat{a} + \hat{b}\cdot\hat{b}$

$= |\hat{a}|^2 - \hat{a}\cdot\hat{b} - \hat{a}\cdot\hat{b} + |\hat{b}|^2$ $\quad (\because |\vec{a}|^2 = \vec{a}\cdot\vec{a}$ & $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a})$

$= 1^2 - 2\hat{a}\cdot\hat{b} + 1^2$ $\quad (\because \hat{a}, \hat{b}$ are unit vectors)

$= 2 - 2\hat{a}\cdot\hat{b} = 2 - 2|\hat{a}||\hat{b}|\cos\theta$ $\quad (\theta =$ Angle bet^n $\hat{a}$ and $\hat{b})$

$= 2 - 2\times1\times1\cos\theta = 2(1-\cos\theta)$

$= 2\times 2\sin^2\left(\frac{\theta}{2}\right) = 4\sin^2\left(\frac{\theta}{2}\right)$

$\Rightarrow \sin^2\left(\frac{\theta}{2}\right) = \frac{1}{4}|\hat{a}-\hat{b}|^2$

$\Rightarrow \sin\left(\frac{\theta}{2}\right) = \frac{1}{2}|\hat{a}-\hat{b}|$ ▭

(22) If sum of two unit vectors is unit vectors then show that difference of their magnitude is $\sqrt{3}$.

Sol^n: Let $\hat{a}, \hat{b}, \hat{c}$ are three unit vectors s.t $\hat{a}+\hat{b} = \hat{c}$ —— ①

We have to prove $|\hat{a}-\hat{b}| = \sqrt{3}$. —— ②

Here $|\hat{a}| = |\hat{b}| = |\hat{c}| = 1$.

Now $|\hat{a}+\hat{b}|^2 = |\hat{c}|^2$ From ①

$\Rightarrow (\hat{a}+\hat{b})\cdot(\hat{a}+\hat{b}) = 1 \Rightarrow \hat{a}\cdot\hat{a} + \hat{a}\cdot\hat{b} + \hat{b}\cdot\hat{a} + \hat{b}\cdot\hat{b} = 1$

$\Rightarrow |\hat{a}|^2 + 2\hat{a}\cdot\hat{b} + |\hat{b}|^2 = 1 \Rightarrow 1^2 + 2|\hat{a}||\hat{b}|\cos\theta + 1^2 = 1$

(Let θ = angle between $\hat{a}$ and $\hat{b}$)

$\Rightarrow 2 + 2\times1\times1\cos\theta = 1 \Rightarrow 2 + 2\cos\theta = 1 \Rightarrow 2\cos\theta = -1 \Rightarrow \cos\theta = -\frac{1}{2}$ —— ③

Now $|\hat{a}-\hat{b}|^2 = (\hat{a}-\hat{b})\cdot(\hat{a}-\hat{b}) = \hat{a}\cdot\hat{a} - \hat{a}\cdot\hat{b} - \hat{b}\cdot\hat{a} + \hat{b}\cdot\hat{b}$

$= |\hat{a}|^2 - 2\hat{a}\cdot\hat{b} + |\hat{b}|^2 = 1^2 - 2|\hat{a}||\hat{b}|\cos\theta + 1^2$

$= 2 - 2\times1\times1\left(-\frac{1}{2}\right) = 2+1 = 3$

$\Rightarrow |\hat{a}-\hat{b}| = \sqrt{3}$ ▭

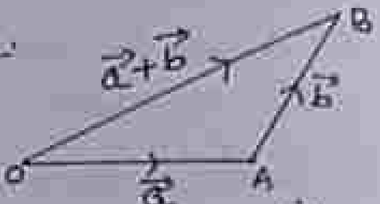
23) Prove that (i) $|\vec{a}+\vec{b}| \le |\vec{a}|+|\vec{b}|$

(ii) $|\vec{a}-\vec{b}| \le |\vec{a}|+|\vec{b}|$

(iii) $|\vec{a}-\vec{b}| \geq |\vec{a}|-|\vec{b}|$

Soln

(i) Let $\vec{a}$ and $\vec{b}$ are not co-linear and $\vec{OA}=\vec{a}, \vec{AB}=\vec{b}$ fig(i)

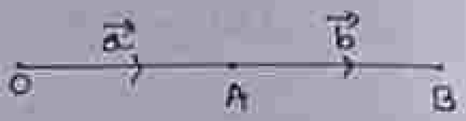


(i)

Then by triangle law of vector addition

$\vec{OB} = \vec{OA}+\vec{AB}$

$\Rightarrow \vec{OB} = \vec{a}+\vec{b}$



(ii)

But from properties of triangle, In triangle OAB

$$OA + AB > OB$$

$$\Rightarrow |\vec{OA}| + |\vec{AB}| > |\vec{OB}|$$

$$\Rightarrow |\vec{a}|+|\vec{b}| > |\vec{a}+\vec{b}|$$

$$\Rightarrow |\vec{a}+\vec{b}| < |\vec{a}|+|\vec{b}| \text{ ——— (1)}$$

But If $\vec{a}$ and $\vec{b}$ are co-linear fig-(ii) then.

$OB = OA + AB$

$\Rightarrow |\vec{OB}| = |\vec{OA}| + |\vec{AB}| \Rightarrow |\vec{a}+\vec{b}| = |\vec{a}|+|\vec{b}|$ ——— (2)

From (1) and (2) we have

$$|\vec{a}+\vec{b}| \le |\vec{a}|+|\vec{b}| \quad \square$$

(ii) $|\vec{a}-\vec{b}| = |\vec{a}+(-\vec{b})| \le |\vec{a}| + |-\vec{b}|$ $\quad \because |\vec{a}+\vec{b}| \le |\vec{a}|+|\vec{b}|$

$= |\vec{a}|+|\vec{b}|$ $\quad \because |\vec{b}| = |-\vec{b}|$

$\Rightarrow |\vec{a}-\vec{b}| \le |\vec{a}|+|\vec{b}| \quad \square$

(iii) $|\vec{a}| = |(\vec{a}-\vec{b})+\vec{b}| \le |\vec{a}-\vec{b}|+|\vec{b}|$ $\quad \because |\vec{a}+\vec{b}| \le |\vec{a}|+|\vec{b}|$

$\Rightarrow |\vec{a}| \le |\vec{a}-\vec{b}|+|\vec{b}|$

$\Rightarrow |\vec{a}|-|\vec{b}| \le |\vec{a}-\vec{b}|$

$\Rightarrow |\vec{a}-\vec{b}| \geq |\vec{a}|-|\vec{b}| \quad \square$.

(24) If $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 3\hat{i} + 4\hat{j} - \hat{k}$ then find the sine of the angle between them

Solⁿ: $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = 3\hat{i} + 4\hat{j} - \hat{k}$

Let θ be the angle between $\vec{a}$ and $\vec{b}$. We have to find out $\sin\theta$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix} = \hat{i}(1-4) - \hat{j}(-2-3) + \hat{k}(8+3)$$

$$= -3\hat{i} + 5\hat{j} + 11\hat{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{(-3)^2 + (5)^2 + (11)^2} = \sqrt{9 + 25 + 121} = \sqrt{155}$$

$$|\vec{a}| = \sqrt{(2)^2 + (-1)^2 + (1)^2} = \sqrt{6}$$

$$|\vec{b}| = \sqrt{(3)^2 + (4)^2 + (-1)^2} = \sqrt{26}$$

Now $$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \frac{\sqrt{155}}{\sqrt{6} \cdot \sqrt{26}} = \sqrt{\frac{155}{156}}$$ (Ans).

(25) Show that the vectors $\hat{i} - 3\hat{j} + 4\hat{k}$, $2\hat{i} - \hat{j} + 2\hat{k}$ and $4\hat{i} - 7\hat{j} + 10\hat{k}$ are co-planar.

Solⁿ $\vec{a} = \hat{i} - 3\hat{j} + 4\hat{k}$

$\vec{b} = 2\hat{i} - \hat{j} + 2\hat{k}$

$\vec{c} = 4\hat{i} - 7\hat{j} + 10\hat{k}$

Now $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 4 \\ 2 & -1 & 2 \\ 4 & -7 & 10 \end{vmatrix} = 1(-10+14) + 3(20-8) + 4(-14+4)$$

$$= 4 + 36 - 40 = 40 - 40 = 0$$

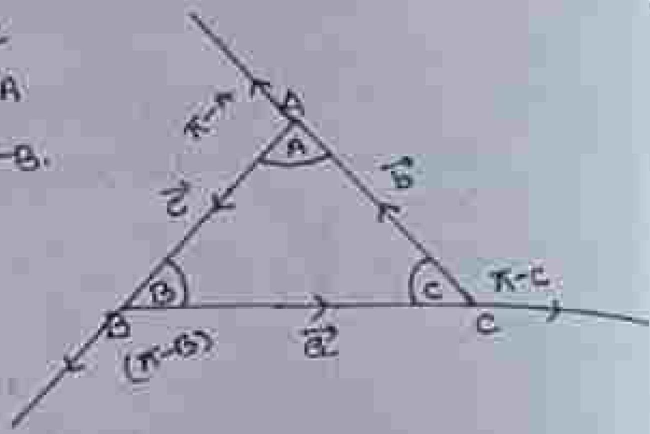
Hence $\vec{a}, \vec{b}, \vec{c}$ are co-planar.

(26) In triangle ABC show that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ where $BC = a$, $CA = b$, $AB = c$ by vector method.

Solⁿ. Let ABC be any triangle having $\vec{a} = \vec{BC}$, $\vec{b} = \vec{CA}$

$\vec{c} = \vec{AB}$

Angle between $\vec{a}$ and $\vec{b} = \pi - C$
Angle between $\vec{b}$ and $\vec{c} = \pi - A$
Angle between $\vec{c}$ and $\vec{a} = \pi - B$.



Now by triangle law of vector addition

$$\vec{BC} + \vec{CA} = \vec{BA} = -\vec{AB}$$

$$\Rightarrow \vec{BC} + \vec{CA} + \vec{AB} = \vec{0}$$

$$\Rightarrow \vec{a} + \vec{b} + \vec{c} = \vec{0} \quad \text{——— (1)}$$

Taking cross product on both sides of (1) by $\vec{a}$,

$$\vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \times \vec{0}$$

$$\Rightarrow \vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0}$$

$$\Rightarrow \vec{0} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0}$$

$$\Rightarrow \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = 0 \Rightarrow \vec{a} \times \vec{b} = -\vec{a} \times \vec{c}$$

$$\Rightarrow \vec{a} \times \vec{b} = \vec{c} \times \vec{a} \quad \text{—— (2)}$$

Similarly taking cross product of both sides of (1) by $\vec{b}$ we have

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} \quad \text{—— (3)}$$

From eqn (2) and (3) we have

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$$

$$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{b} \times \vec{c}| = |\vec{c} \times \vec{a}|$$

$$\Rightarrow |\vec{a}||\vec{b}| \sin(\pi - C) = |\vec{b}||\vec{c}| \sin(\pi - A) = |\vec{c}||\vec{a}| \sin(\pi - B)$$

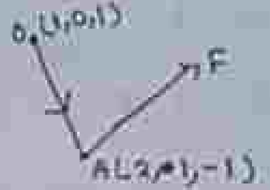
$$\Rightarrow ab \sin(\pi - C) = bc \sin(\pi - A) = ca \sin(\pi - B)$$

$$\Rightarrow \frac{ab \sin C}{abc} = \frac{bc \sin A}{abc} = \frac{ca \sin B}{abc}$$

$$\Rightarrow \frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\Rightarrow \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \square$$

Ex-27: Find the moment about (1,0,1) of the force $2\hat{i}+3\hat{j}+5\hat{k}$ acting at (2,1,-1).



Soln: Force $\vec{F} = 2\hat{i}+3\hat{j}+5\hat{k}$

points are O(1,0,1) and A(2,1,-1)

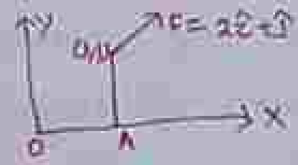
$\vec{r} = \vec{OA} = (2-1)\hat{i}+(1-0)\hat{j}+(-1-1)\hat{k}$

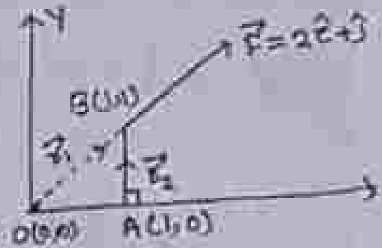
$= \hat{i}+\hat{j}-2\hat{k}$

Moment about O $= \vec{r}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -2 \\ 2 & 3 & 5 \end{vmatrix}$

$= \hat{i}(5+6)-\hat{j}(5+4)+\hat{k}(3-2) = 11\hat{i}-9\hat{j}+\hat{k}$ (Ans)

Ex-28 Find the torque about point O and A due to given force $\vec{F}$ from the figure





Soln: From figure

$\vec{F} = 2\hat{i}+\hat{j}$ acts at B(1,1)

points are O(0,0), A(1,0)

$\vec{r_1} = \vec{OB} = (1-0)\hat{i}+(1-0)\hat{j} = \hat{i}+\hat{j}$

$\vec{r_2} = \vec{AB} = (1-1)\hat{i}+(1-0)\hat{j} = \hat{j}$.

Moment about O $= M_1 = \vec{r_1}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{vmatrix} = \hat{i}(0-0)-\hat{j}(0-0)+\hat{k}(1-2)$

$= -\hat{k}$

Moment about A $= M_2 = \vec{r_2}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ 2 & 1 & 0 \end{vmatrix} = 1(0-2\hat{k}) = -2\hat{k}$

$\therefore$ Moment about O $= -\hat{k}$ and about A is $-2\hat{k}$ (Ans)

29. An object has an angular speed of 4 rad/sec and the axis of rotation passes through the points (1,2,1) and (2,2,-1). Find velocity of particle at M(2,1,3)

Soln. Linear velocity $= \vec{V}$

angular velocity $= \vec{\omega}$

Axis of rotation is PQ

$|\vec{\omega}| = 4$ rad./sec

P(1,2,1), Q(2,2,-1), M(2,1,3) are three points. $\vec{r}$ = position vector of M w.r.t P

$= \vec{PM} = (2-1)\hat{i}+(1-2)\hat{j}+(3-1)\hat{k}$

$= \hat{i}-\hat{j}+2\hat{k}$

$\vec{PQ} = (2-1)\hat{i}+(2-2)\hat{j}+(-1-1)\hat{k}$

$= \hat{i}-2\hat{k}$

unit vector along $\vec{PQ} = \frac{\vec{PQ}}{|\vec{PQ}|} = \frac{\hat{i}-2\hat{k}}{\sqrt{(1)^2+(-2)^2}} = \frac{\hat{i}-2\hat{k}}{\sqrt{5}}$

$\vec{\omega} = |\vec{\omega}| \times$ unit vector along $\vec{PQ}$

$= 4 \times \frac{\hat{i}-2\hat{k}}{\sqrt{5}} = \frac{4}{\sqrt{5}}(\hat{i}-2\hat{k})$

velocity at M $\vec{v} = \vec{\omega} \times \vec{r}$

$= \frac{4}{\sqrt{5}}(\hat{i}-2\hat{k}) \times (\hat{i}-\hat{j}+2\hat{k})$

$= \frac{4}{\sqrt{5}} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2 \\ 1 & -1 & 2 \end{vmatrix} = \frac{4}{\sqrt{5}} \{ \hat{i}(0-2) - \hat{j}(2+2) + \hat{k}(-1-0) \}$

$= \frac{4}{\sqrt{5}}(-2\hat{i}-4\hat{j}-\hat{k}) = -\frac{4}{\sqrt{5}}(2\hat{i}+4\hat{j}+\hat{k})$ (Ans)

30. If vectors $\vec{P}, \vec{Q}, \vec{R}$ have magnitude 5, 12, 13 units and $\vec{P}+\vec{Q}=\vec{R}$ then find angle between $\vec{Q}, \vec{R}$.

Soln $|\vec{P}| = 5, |\vec{Q}| = 12, |\vec{R}| = 13$

$\because \vec{P}+\vec{Q}=\vec{R} \Rightarrow \vec{R}$ is the resultant of $\vec{P}$ and $\vec{Q}$

$\Rightarrow \vec{P}, \vec{Q}$ are in same order and $\vec{R}$ in opposite order.

Here $(AB)^2+(BC)^2 = 5^2+12^2 = 169 = 13^2 = AC^2$

$\therefore$ ABC is a right angle triangle.

Let θ be the angle between $\vec{R}$ and $\vec{Q}$

then $\cos\theta = \frac{BC}{AC} = \frac{|\vec{Q}|}{|\vec{R}|} = \frac{12}{13}$

$\Rightarrow \theta = \cos^{-1}\frac{12}{13}$ (ANS)

Scanned with CamScanner

EXERCISE

(1) show that the vectors $\vec{a} = 3\sqrt{3}\hat{i} - 3\hat{j}$, $\vec{b} = 6\hat{j}$, $\vec{c} = 3\sqrt{3}\hat{i} + 3\hat{j}$ forms the sides of an equilateral Triangle

(2) Find unit vector in the direction of vector $\vec{a} = 3\hat{i} - 4\hat{j} + \hat{k}$.

(3) Find unit vector in the direction of $\vec{a}+\vec{b}$ where $\vec{a} = \hat{i}+\hat{j}-\hat{k}$ and $\vec{b} = \hat{i}-\hat{j}+3\hat{k}$.

(4) Find the value of p for which the vectors $\vec{a} = 3\hat{i}-\hat{j}+\hat{k}$ and $\vec{b} = p\hat{i}+\hat{j}+3\hat{k}$ are perpendicular.

(5) Find 'p' for which the vectors $2\hat{i}+3\hat{j}-p\hat{k}$ and $4\hat{i}+6\hat{j}+3\hat{k}$ are parallel to each other.

(6) Find scalar product of $3\hat{i}-4\hat{j}$ and $-2\hat{i}+\hat{j}$.

(7) Find the angle between the vectors $5\hat{i}+\hat{j}-3\hat{k}$ and $3\hat{i}-6\hat{j}+\hat{k}$

(8) Find the scalar and vector projection of $\vec{a}$ on $\vec{b}$ where $\vec{a} = \hat{i}-\hat{j}-\hat{k}$ and $\vec{b} = 3\hat{i}+\hat{j}+3\hat{k}$

(9) Find scalar and vector projection of $\vec{b}$ on $\vec{a}$ where $\vec{a} = 3\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b} = 2\hat{i}+3\hat{j}-4\hat{k}$.

(10) For any two vectors $\vec{a}$ and $\vec{b}$ prove that $(\vec{a}\times\vec{b})^2 = a^2b^2 - (\vec{a}\cdot\vec{b})^2$

(11) Find the workdone by the force $\vec{F} = \hat{i}+\hat{j}-\hat{k}$ acting on a particle if the particle is displaced from the point A(3,3,3) to B(4,4,4).

(12) calculate Area of triangle ABC having vertices A(1,1,2), B(2,2,3), and C(3,-1,-1) by vector method.

(13) Find area of the parallelogram whose vertices are A(5,-1,1), B(-1,-3,4), C(1,-6,10), D(7,-4,7).

(14) If $\vec{a} = \hat{i}+3\hat{j}-2\hat{k}$ and $\vec{b} = -\hat{i}+3\hat{k}$ then find $|\vec{a}\times\vec{b}|$

(15) Determine the area of the parallelogram whose adjacent sides are the vectors $\vec{a} = 2\hat{i}$ and $\vec{b} = 3\hat{j}$.

(16) Find the unit vector perpendicular to both the vectors $\vec{a} = 2\hat{i}+\hat{j}-\hat{k}$ and $\vec{b} = 3\hat{i}-\hat{j}+3\hat{k}$.

(17) Prove by vector method, in any triangle ABC,

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$ where $BC = a$, $CA = b$, $AB = c$

(18) If $\vec{F} = 2\hat{i}$, $\vec{r} = 3\hat{j}$ then find moment of force. 26

(19) If angular velocity $\omega = 3\hat{k}$, radius $\hat{r} = 3\hat{j}$ then find linear velocity

(20) Find the torque of the force $\vec{F} = (2\hat{i} - 3\hat{j} + 4\hat{k})$ N acting at point $\vec{r} = (3\hat{i} + 2\hat{j} + 3\hat{k})$ m.

A:
DIFFERENTIAL EQUATION: (D.E)
→ A differential equation is an equation involving dependent variable independent variables and derivatives of dependent variable w.r.t one or more independent variables.

→ Ex: $\frac{dy}{dx} + x^2 = 2$ (y = Dependent variable (D.V), x = Independent Variable (I.V)).

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ (u = D.V, x, y, z = I.V,s).

A1: ORDINARY DIFFERENTIAL EQUATION (O.D.E)

→ The differential equation which contains only one independent variable, one dependent variable and derivatives of dependent variable w.r.t independent variable is called ordinary differential equation.

→ $\frac{dy}{dx} + x^2 = 2$ is an O.D.E (y = D.V, x = I.V)

A2: PARTIAL DIFFERENTIAL EQUATION: (P.D.E)
→ The differential equation which contains more than one independent variables, one dependent variable and partial derivatives of dependent variable w.r.t independent variables is called partial differential equation.

→ $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ is a P.D.E (u = D.V, x, y, z = I.V S)

A3: ORDER OF D.E → The order of the highest order derivative occuring in the differential equation, is called order of the equation

→ Ex: $\frac{d^3 y}{dx^3} + 2x\frac{d^2 y}{dx^2} + \frac{dy}{dx} = \sin x$. its order = 3.

A4: DEGREE OF D.E → The highest exponent of the highest order derivative after removing radicals and fractions from derivatives involving in the equation is called degree of the differential equation.

→ Ex: degree of $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^4 + x$ is 1

A₃: Linear and Non-linear D.E:

→ A Differential equation in which
(i) Every dependent variable and its derivatives have power 1.
(ii) Dependent variable and derivatives or two derivatives terms are not multiplied together
is called linear differential equation otherwise it is called Non-linear.

→ Example:

D. E	Linear / Non Linear
$\frac{dy}{dx} + xy = x^2$	Linear
$\frac{d^3y}{dx^3} + x^4 \frac{d^2y}{dx^2} + y = \sin x$	Linear
$\frac{d^3y}{dx^3} + \underline{y \frac{d^2y}{dx^2}} = 4x$	Non Linear
$\underline{\left(\frac{dy}{dx}\right)^3} + 1 = 2\frac{dy}{dx}$	Non-linear
$\frac{d^3y}{dx^3} + \underline{\frac{dy}{dx} \cdot \frac{d^2y}{dx^2}} + y = 4x^2$	Non-linear

For presence of underline term.

B: SOLUTION (PRIMITIVE)

→ A functional relationship (Explicit or implicit) between dependent and independent variables satisfying the differential equation is called solution of differential equation.

→ $y = f(x)$ or $f(x,y) = 0$ will represent a solution of D.E
$F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2} \ldots \frac{d^ny}{dx^n}) = 0$

B₁: GENERAL SOLUTION:

→ The solution of differential equation containing as many arbitrary constants as the order of the differential equation is called as the general solution.

→ $y = \sin x + c$ is the general solution of $\frac{dy}{dx} - \cos x = 0$
c = arbitrary constant, order of D.E is 1.

B:
PERTICULAR SOLUTION:

→ perticular solution is a solution is is obtained by assigning perticular values to arbitrary constants involved in general solution.

→ $y = \sin x + c$ is the general solution of $\frac{dy}{dx} - \cos x = 0$ —— ①

Hence $y = \sin x$, $y = \sin x + 1$, $y = \sin x - \frac{3}{2}$ ...... are perticular solutions of ① (Taking $c = 0, 1, -\frac{3}{2}$ ......).

C: SOLUTION OF 1st. ORDER, 1st. degree D.E by 'VARIABLE SEPARABLE' METHOD

FORM-I $\boxed{\frac{dy}{dx} = f(x)}$

process: $\frac{dy}{dx} = f(x) \Rightarrow dy = f(x)dx$ (variable separated)

$\Rightarrow \int dy = \int f(x)dx + c \Rightarrow y = \phi(x) + c$ where $\int f(x)dx = \phi(x)$ (Say)

FORM-II $\boxed{\frac{dy}{dx} = f(y)}$

process: $\frac{dy}{dx} = f(y) \Rightarrow dy = f(y)\cdot dx \Rightarrow \frac{dy}{f(y)} = dx$ (variable separated)

$\Rightarrow \int \frac{dy}{f(y)} = \int dx + c \Rightarrow \phi(y) = x + c$ where $\int \frac{dy}{f(y)} = \phi(y)$ (Say)

FORM-III $\boxed{f(x)dy + g(y)dx = 0}$

process: $f(x)dy + g(y)dx = 0 \Rightarrow \frac{f(x)dy}{f(x)\cdot g(y)} + \frac{g(y)dx}{f(x)\cdot g(y)} = 0$

$\Rightarrow \frac{dy}{g(y)} + \frac{dx}{f(x)} = 0$ (variable separated).

$\Rightarrow \int \frac{dy}{g(y)} + \int \frac{dx}{f(x)} = \int 0 \cdot dx + c \Rightarrow \phi(y) + \psi(x) = c$ where

$\phi(y) = \int \frac{dy}{g(y)}$, $\psi(x) = \int \frac{dx}{f(x)}$

FORM-IV $\boxed{f_1(x)\, g_1(y)dx + f_2(x)\, g_2(y)dy = 0}$

process: Divide the eqn by $f_2(x)\cdot g_1(y)$ and do like (III)

FORM-V: $\boxed{\frac{dy}{dx} = f(ax + by + c)}$

process: put $ax + by + c = z$ and proceed.

C: SOLUTION OF LINEAR DIFFERENTIAL EQUATION OF FOLLOWING FORMS:

FORM-I: $\boxed{\frac{dy}{dx} + Py = Q}$, P, Q are functions of x.

SOLUTION OF LINEAR DIFFERENTIAL EQUATION:

FORMS	INTEGRATING FACTOR (I.F)	GENERAL SOLUTION
① $\frac{dy}{dx} + Py = Q$ $P, Q \to x$	$I.F = e^{\int P dx}$	$y \times I.F = \int Q \times (I.F) dx + c$
②. $\frac{dx}{dy} + Px = Q$ $P, Q \to y$	$I.F = e^{\int P . dy}$	$x \times I.F = \int Q \times (I.F) dy + c$

PROBLEMS:

1. Find order and degree of following differential equations.

(a) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{5/2} = 3\frac{d^2y}{dx^2}$

(b) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 1 = 0$ ⓒ $\sqrt{1-y^2}dx + y\sqrt{1-x^2}dy = 0$

Solⁿ
(a) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{5/2} = 3\frac{d^2y}{dx^2} \Rightarrow \left\{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{5}{2}}\right\}^2 = \left(3 \cdot \frac{d^2y}{dx^2}\right)^2$

$\Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right]^5 = 9 \cdot \left(\frac{d^2y}{dx^2}\right)^2$ ——①

Highest order derivative is $\left(\frac{d^2y}{dx^2}\right)$ whose power is 2

Hence order = 2, degree = 2 (Ans)

ⓑ $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 1 = 0$ ——①

Highest order derivative is $\frac{d^2y}{dx^2}$ whose power is 1

∴ order = 2, degree = 1 (Ans)

ⓒ $\sqrt{1-y^2}dx + y\sqrt{1-x^2}dy = 0 \Rightarrow y\sqrt{1-x^2}dy = -\sqrt{1-y^2}dx$

$\Rightarrow \frac{dy}{dx} = \frac{-\sqrt{1-y^2}}{y\sqrt{1-x^2}}$ ——①

Highest order derivative is $\frac{dy}{dx}$ whose power is 1.

∴ order = 1, degree = 1. (Ans)

2: Solve $\frac{d^2y}{dx^2} = 3$

Solⁿ: $\frac{d^2y}{dx^2} = 3 \Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = 3 \Rightarrow d\left(\frac{dy}{dx}\right) = 3dx$

D: FORMATION OF DIFFERENTIAL EQUATION:

A General Solution of a differential equation is given. We have to form the differential equation corresponding to it. A differential equation can be formed by eliminating the arbitrary constant(s) involved in the solution equation by following steps:

→ Write down the general solution containing arbitrary constants and named it as equation ①

→ Differentiate eqⁿ(1) multiple times as required and named these as eqⁿ(2),(3)- - - - w.r.t the independent variable

→ Eliminate the arbitrary constants involved by combining Eqⁿ(1),(2),(3) - - - into a single equation, resulting the ordinary differential equation.

PROBLEMS:

1. Find order and degree of following differential equations

(a) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{5/2} = 3\frac{d^2y}{dx^2}$

(b) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 1 = 0$ (c) $\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0$

Soln

(a) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{5/2} = 3\frac{d^2y}{dx^2} \Rightarrow \left\{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{5}{2}}\right\}^2 = \left(3\cdot\frac{d^2y}{dx^2}\right)^2$

$\Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right]^5 = 9\cdot\left(\frac{d^2y}{dx^2}\right)^2$ ——— (1)

Highest order derivative is $\left(\frac{d^2y}{dx^2}\right)$ whose power is 2

Hence order = 2, degree = 2 (Ans)

(b) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 1 = 0$ — (1)

Highest order derivative is $\frac{d^2y}{dx^2}$ whose power is 1

$\therefore$ order = 2, degree = 1 (Ans)

(c) $\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0 \Rightarrow y\sqrt{1-x^2}\,dy = -\sqrt{1-y^2}\,dx$

$\Rightarrow \frac{dy}{dx} = \frac{-\sqrt{1-y^2}}{y\sqrt{1-x^2}}$ ——— (1)

Highest order derivative is $\frac{dy}{dx}$ whose power is 1.

$\therefore$ order = 1, degree = 1. (Ans)

2. Solve $\frac{d^2y}{dx^2} = 3$

Soln: $\frac{d^2y}{dx^2} = 3 \Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = 3 \Rightarrow d\left(\frac{dy}{dx}\right) = 3\,dx$

$\Rightarrow \int d\left(\frac{dy}{dx}\right) = \int 3dx \Rightarrow \frac{dy}{dx} = 3x + c_1 \Rightarrow dy = (3x+c_1)dx$

$\Rightarrow \int dy = \int (3x+c_1)dx = 3\int x dx + c_1\int dx$

$\Rightarrow y = 3 \times \frac{x^2}{2} + c_1 x + c_2 = \frac{3}{2}x^2 + c_1 x + c_2$ (Ans)

3. Solve: $\frac{dy}{dx} = \frac{1}{1+x^2}$

Soln: $\frac{dy}{dx} = \frac{1}{1+x^2} \Rightarrow dy = \frac{1}{1+x^2}dx \Rightarrow \int dy = \int \frac{1}{1+x^2}dx$

$\Rightarrow y = \tan^{-1}x + c$ (Ans).

4: Solve: $\frac{dy}{dx} = \cot^2 y$.

Soln: $\frac{dy}{dx} = \cot^2 y \Rightarrow dy = \cot^2 y\, dx \Rightarrow \frac{dy}{\cot^2 y} = dx$

$\Rightarrow \tan^2 y \cdot dy = dx \Rightarrow \int \tan^2 y \cdot dy = \int dx \Rightarrow \int (\sec^2 y - 1)dy = x + c$

$\Rightarrow \int \sec^2 y\, dy - \int dy = x + c \Rightarrow \tan y - y = x + c$

$\Rightarrow \tan y - x - y = c$ (Ans)

5: Solve: $\frac{dy}{dx} + \frac{1+\cos 2y}{1-\cos 2x} = 0$

Soln: $\frac{dy}{dx} + \frac{1+\cos 2y}{1-\cos 2x} = 0 \Rightarrow \frac{dy}{dx} + \frac{2\cos^2 y}{2\sin^2 x} = 0 \Rightarrow \frac{dy}{dx} + \frac{\cos^2 y}{\sin^2 x} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{\cos^2 y}{\sin^2 x} \Rightarrow \sin^2 x = -\cos^2 y \cdot dx$

$\Rightarrow \sin^2 x \cdot dy + \cos^2 y\, dx = 0 \Rightarrow \frac{\sin^2 x}{\sin^2 x \cdot \cos^2 y}dy + \frac{\cos^2 y}{\sin^2 x \cdot \cos^2 y}dx = 0$

$\Rightarrow \sec^2 y \cdot dy + \operatorname{cosec}^2 x\, dx = 0 \Rightarrow \int (\sec^2 y\, dy + \operatorname{cosec}^2 x\, dx) = \int 0\, dx$

$\Rightarrow \int \sec^2 y\, dy + \int \operatorname{cosec}^2 x\, dx = 0 + c$

$\Rightarrow \tan y - \cot x = c$ (ANS)

6: Solve: $x(1+y^2)dx + y(1+x^2)dy = 0$

Soln: $x(1+y^2)dx + y(1+x^2)dy = 0$

$\Rightarrow \frac{x(1+y^2)dx}{(1+x^2)(1+y^2)} + \frac{y(1+x^2)dy}{(1+x^2)(1+y^2)} = 0$

$\Rightarrow \frac{x\,dx}{1+x^2} + \frac{y\,dy}{1+y^2} = 0 \Rightarrow \int\left[\frac{x\,dx}{1+x^2} + \frac{y\,dy}{1+y^2}\right] = \int 0 \cdot dx$

$\Rightarrow \int \frac{x\,dx}{1+x^2} + \int \frac{y\,dy}{1+y^2} = 0 + \ln c \Rightarrow \frac{1}{2}\int \frac{dt}{t} + \frac{1}{2}\int \frac{dz}{z} = \frac{\ln c}{2}$

(putting $1+x^2 = t$, $1+y^2 = z$)

$\Rightarrow \frac{1}{2}\ln t + \frac{1}{2}\ln z = \frac{\ln c}{2} \Rightarrow \ln(1+x^2) + \ln(1+y^2) = \ln c$

$\Rightarrow \ln(1+x^2)(1+y^2) = \ln c \Rightarrow (1+x^2)(1+y^2) = c$ (Ans)

7: Solve $x\log x \frac{dy}{dx} + y = 2\log x$.

Soln: $x\log x \frac{dy}{dx} + y = 2\log x$

$\Rightarrow \frac{dy}{dx} + \frac{y}{x\log x} = \frac{2}{x}$ Dividing both sides by $x\log x$.

$\Rightarrow \frac{dy}{dx} + \frac{1}{x\log x} y = \frac{2}{x}$ ——(1) which is linear

Here $P = \frac{1}{x\log x}$, $Q = \frac{2}{x}$

Integrating factor $I.F = e^{\int P dx} = e^{\int \frac{1}{x\log x} dx}$

$= e^{\int \frac{dt}{t}}$ (putting $\log x = t$)

$= e^{\ln t} = t = \log x$

General solution is, $y \times I.F = \int Q \times (I.F)\, dx + c$

$\Rightarrow y \times \log x = \int \frac{2}{x} \times \log x \, dx + c$

$\Rightarrow y \log x = 2\int (\log x) \cdot (\frac{1}{x} dx) + c$

$\Rightarrow y \log x = 2\int z \cdot dz + c$ putting $\log x = z$

$= 2 \times \frac{z^2}{2} + c = z^2 + c = (\log x)^2 + c$

$\Rightarrow y \log x - (\log x)^2 = c \Rightarrow (y - \log x)\log x = c$ (Ans)

Q: Solve: $\cos(x+y)dy = dx$

Sol: $\cos(x+y)dy = dx \Rightarrow \frac{dy}{dx} = \frac{1}{\cos(x+y)}$ ——(1)

put $x + y = z \Rightarrow \frac{d}{dx}(x+y) = \frac{dz}{dx}$

$\Rightarrow 1 + \frac{dy}{dx} = \frac{dz}{dx} \Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$ ——(2)

From eqn (1) & eqn (2),

$\frac{dz}{dx} - 1 = \frac{1}{\cos(x+y)} \Rightarrow \frac{dz}{dx} - 1 = \frac{1}{\cos z} \Rightarrow \frac{dz}{dx} = \frac{1}{\cos z} + 1$

$\Rightarrow \frac{dz}{dx} = \frac{1+\cos z}{\cos z} \Rightarrow \cos z \, dz = (1+\cos z)dx$

$\Rightarrow \frac{\cos z}{1+\cos z} dz = dx \Rightarrow \int \frac{\cos z}{(1+\cos z)} dz = \int dx + c$

$\Rightarrow \int \frac{(1+\cos z) - 1}{(1+\cos z)} dz = x + c \Rightarrow \int \left(1 - \frac{1}{1+\cos z}\right) dz = x + c$

$\Rightarrow \int dz - \int \frac{1}{1+\cos z} dz = x + c \Rightarrow z - \int \frac{1}{2\cos^2(\frac{z}{2})} dz = x + c$

$\Rightarrow z - \frac{1}{2}\int \sec^2(z/2)\, dz = x + c$

$\Rightarrow z - \frac{1}{2}\int \sec^2 t \cdot 2dt = x + c$ | put $\frac{z}{2} = t \Rightarrow \frac{1}{2}dz = dt \Rightarrow dz = 2dt$

$\Rightarrow z - \tan t = x + c \Rightarrow x + y - \tan(\frac{z}{2}) = x + c$

$\Rightarrow y - \tan\left(\frac{x+y}{2}\right) = c$ (Ans).

Ex-9: Form the differential equation representing the family of curves $y = ax$ where a is arbitrary constant.

Sol^n: $y = ax$ ——(1)

$$\frac{dy}{dx} = \frac{d}{dx}(y) = \frac{d}{dx}(ax) = a\frac{d}{dx}(x) = a \times 1 = a$$

$$\therefore \frac{dy}{dx} = a \text{ ——(2)}$$

Using Eqn (2) in (1),

$$y = ax \Rightarrow y = \frac{dy}{dx}.x \Rightarrow x\frac{dy}{dx} - y = 0$$

which is the required solution.

EXERCISE

1. Find order and degree of following Differential equations:

(a) $\frac{dy}{dx}+xy=1$ (b) $\frac{d^2y}{dx^2}=K\left[1+\left(\frac{dy}{dx}\right)^2\right]$ (c) $\frac{dy}{dx}=\frac{3}{(dy/dx)}$

(d) $\frac{d^2y}{dx^2}=\sqrt{y+\frac{dy}{dx}}$ (e) $(xy^2+x)dx+(y-x^2)dy=0$

(f) $y\cot x\,dx+\tan x\,dy=0$ (g) $\left(\frac{dy}{dx}\right)^3+y^3=\frac{d^2y}{dx^2}$

(h) $\log\left(\frac{d^2y}{dx^2}\right)=y$ (i) $\frac{1}{x}\frac{d^2y}{dx^2}=e^x$ (j) $\left(\frac{dy}{dx}\right)^4+\frac{d^3y}{dx^3}=3yx\frac{dy}{dx}$

(k) $\left(\frac{dy}{dx}\right)^4+y^5=\frac{d^3y}{dx^3}$ (l) $\left(\frac{d^3y}{dx^3}+5\frac{d^2y}{dx^2}+6\frac{dy}{dx}\right)^{2/3}+y=0$

2. Solve:

(a) $\frac{d^2y}{dx^2}=0$ (b) $\frac{dy}{dx}=\frac{x}{y}$ (c) $\frac{dy}{dx}=e^x$ (d) $\frac{dy}{dx}=\frac{1-\cos x}{1+\cos x}$

(e) $x^2(y-1)dx+y^2(x-1)dy=0$

(f) $(1+x)(1+y^2)dx+(1+y)(1+x^2)dy=0$

(g) $\sec x\tan y\,dx+\sec^2y\cdot\tan x\,dy=0$

(h) $\frac{dy}{dx}+1=e^{x+y}$

(i) $\frac{dy}{dx}=\sin(x+y)$

(j) $x\log x\frac{dy}{dx}+y=\log x^2$

(k) $\frac{dy}{dx}+y\cot x=\cos x$.

(l) $(1+y^2)dx=(\tan^{-1}y-x)dy$

(m) $\frac{dy}{dx}+(\sec x)y=\tan x$

(n) $\frac{dx}{dy}+\sqrt{\frac{1-x^2}{1-y^2}}=0$

—END—