LECTURER NOTES
ON

STRENGTH OF MATRIAL
(Theory -2)

FOR
3rd Semester Mechanical Engineering
(As per SCTE&VT Syllabus)

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Strength of Material

Chapter- 01

STRENGTH OF MATERIALS

When an external force acts on a body, the body tends to undergo some deformation. Due to cohesion between the molecules, the body resists deformation. The resistance by which material of the body opposes the deformation is known as **strength of material**. Within a certain limit (i.e,the elastic stage) the resistance offered by the material is proportional to the deformation brought out on the material by external force. Also within this limit the resistance is equal to external force or applied load. But beyond the elastic stage the resistance offered by the material is less than the applied load. In such cases deformation continues, until failure takes place.
Within elastic stage, the resisting force per unit area is called stress or Intensity of Stress.

Stress:-

The Force of resistance per unit area, offered by a body against deformation is known as stress. The external force acting on the body is called load or force. The load is applied on the body while the stress is induced in the material of the body. A loaded member remains in equilibrium when the resistance offered by the member against the deformation and the applied load are equal.

Mathematically stress is written as, $\sigma = \frac{P}{A}$

where σ = Stress (also called intensity of stress),

P = External force or load, and

A = Cross-sectional area.

1.2.1. Units of Stress. The unit of stress depends upon the unit of load (or force) and unit of area. In M.K.S. units, the force is expressed in kgf and area in metre square (*i.e.*, m^2). Hence unit of stress becomes as kgf/m^2. If area is expressed in centimetre square (*i.e.*, cm^2), the stress is expressed as kgf/cm^2.

In the S.I. units, the force is expressed in newtons (written as N) and area is expressed as m^2. Hence unit of stress becomes as N/m^2. The area is also expressed in millimetre square then unit of force becomes as N/mm^2

$$1\ N/m^2 = 1\ N/(100\ cm)^2 = 1\ N/10^4\ cm^2$$

$$= 10^{-4}\ N/cm^2 \text{ or } 10^{-6}\ N/mm^2 \qquad \left(\because \frac{1}{cm^2} = \frac{1}{10^2\ mm^2}\right)$$

$\therefore$ $1\ N/mm^2 = 10^6\ N/m^2$.

Also $1\ N/m^2 = 1$ Pascal $= 1$ Pa.

Notes. **1.** Newton is a force acting on a mass of one kg and produces an acceleration of 1 m/s^2 *i.e.*,

$$1\ N = 1\ (kg) \times 1\ m/s^2.$$

2. The stress in S.I. units is expressed in N/m^2 or N/mm^2.

3. The stress 1 N/mm^2 = 10^6 N/m^2 = MN/m^2. Thus one N/mm^2 is equal to one MN/m^2.

4. One pascal is written as 1 Pa and is equal to 1 N/m^2.

Strain:-

When a body is subjected to some external force, there is some change of dimension of the body. The ratio of change of dimension of the body to the original dimension is known as strain. Strain is dimensionless.

Strain may be :

1. Tensile strain,
2. Compressive strain,
3. Volumetric strain, and
4. Shear strain.

If there is some increase in length of a body due to external force, then the ratio of increase of length to the original length of the body is known as *tensile strain*. But if there is some decrease in length of the body, then the ratio of decrease of the length of the body to the original length is known as *compressive strain*. The ratio of change of volume of the body to the original volume is known as *volumetric strain*. The strain produced by shear stress is known as shear strain.

Types of Stresses:-

The Stresses are classified into two types
i) Normal stress or Direct Stress
ii) Shear stress

The Normal Stress is divided into two types
a) Tensile Stress and
b) Compressive Stress

1.4.1. Tensile Stress. The stress induced in a body, when subjected to two equal and opposite pulls as shown in Fig. 1.1 (*a*) as a result of which there is an increase in length, is known as tensile stress. The ratio of increase in length to the original length is known as *tensile strain*. The tensile stress acts normal to the area and it pulls on the area.

Let P = Pull (or force) acting on the body,
A = Cross-sectional area of the body,
L = Original length of the body,
dL = Increase in length due to pull P acting on the body,
σ = Stress induced in the body, and
e = Strain (*i.e.*, tensile strain).

Fig. 1.1 (*a*) shows a bar subjected to a tensile force P at its ends. Consider a section x-x, which divides the bar into two parts. The part left to the section x-x, will be in equilibrium if P = Resisting force (R). This is shown in Fig. 1.1 (*b*). Similarly the part right to the section x-x, will be in equilibrium if P = Resisting force as shown in Fig. 1.1 (*c*). This resisting force per unit area is known as stress or intensity of stress.

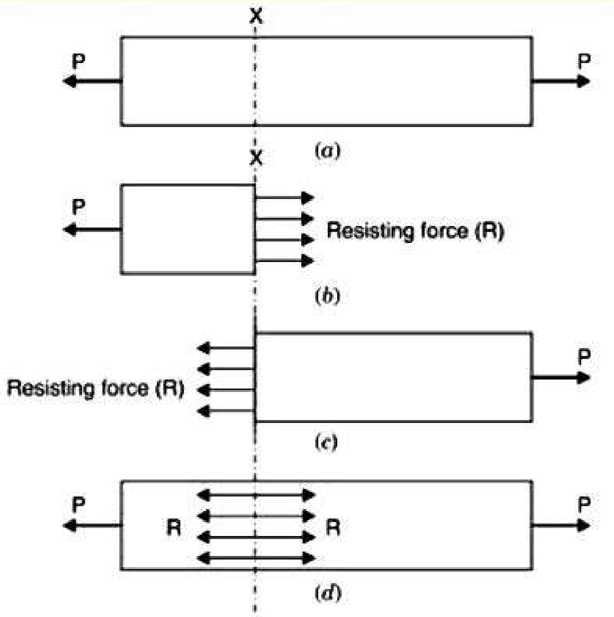


Fig. 1.1

$$\therefore \quad \text{Tensile stress} = \sigma = \frac{\text{Resisting force } (R)}{\text{Cross-sectional area}} = \frac{\text{Tensile load } (P)}{\text{A}} \qquad (\because \quad P = R)$$

$$\sigma = \frac{P}{A} \qquad ...(1.1)$$

And tensile strain is given by,

$$e = \frac{\text{Increase in length}}{\text{Original length}} = \frac{dL}{L}. \qquad ...(1.2)$$

1.4.2. Compressive Stress. The stress induced in a body, when subjected to two equal and opposite pushes as shown in Fig. 1.2 (*a*) as a result of which there is a decrease in length of the body, is known as compressive stress. And the ratio of decrease in length to the original length is known as *compressive strain.* The compressive stress acts normal to the area and it pushes on the area.

Let an axial push P is acting on a body in cross-sectional area A. Due to external push P, let the original length L of the body decreases by dL.

Let P = Force acting on the body
A = Cross sectional Area of the body
L = Original Length of the body
dL = Increase in length of the body due to Force P acting on the body
σ = Stress induced in the body
e = Compressive Strain

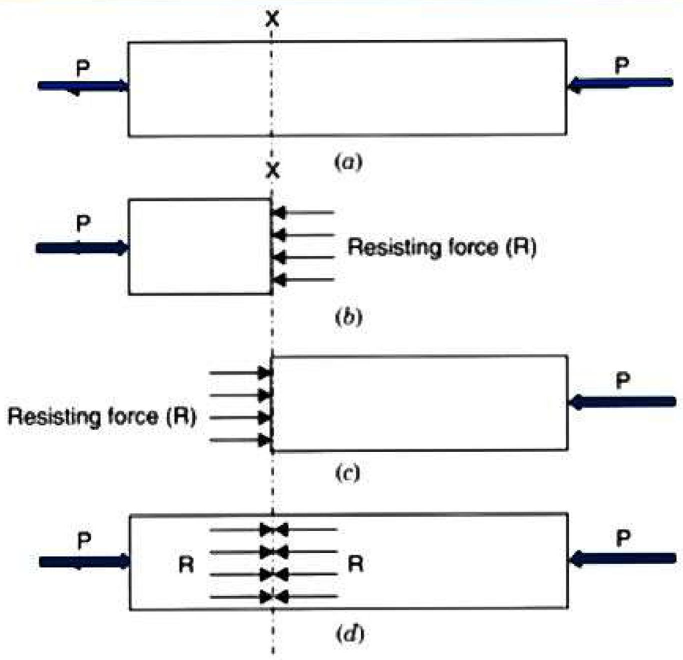


Fig. 1.2

Then compressive stress is given by,

$$\sigma = \frac{\text{Resisting Force }(R)}{\text{Area }(A)} = \frac{\text{Push }(P)}{\text{Area }(A)} = \frac{P}{A}.$$

And compressive strain is given by,

$$e = \frac{\text{Decrease in length}}{\text{Original length}} = \frac{dL}{L}.$$

1.4.3. Shear Stress. The stress induced in a body, when subjected to two equal and opposite forces which are acting tangentially across the resisting section as shown in Fig. 1.3 as a result of which the body tends to shear off across the section, is known as shear stress. The corresponding strain is known as *shear strain*. The shear stress is the stress which acts tangential to the area. It is represented by τ.

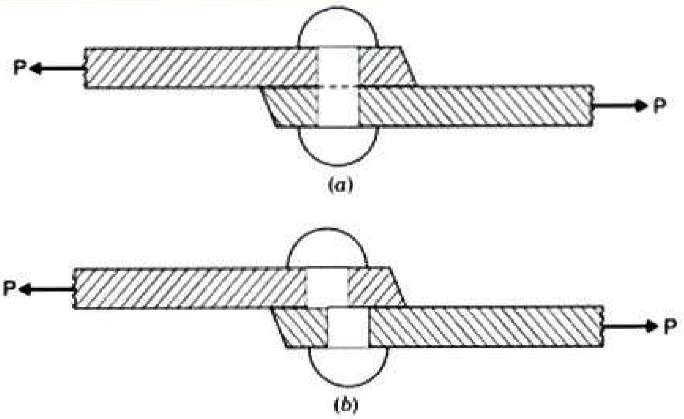


Fig. 1.3

Consider a rectangular block of height h, length L and width unity. Let the bottom face AB of the block be fixed to the surface as shown in Fig. 1.4 (*a*). Let a force P be applied tangentially along the top face CD of the block. Such a force acting tangentially along a surface is known as shear force. For the equilibrium of the block, the surface AB will offer a tangential reaction P equal and opposite to the applied tangential force P.

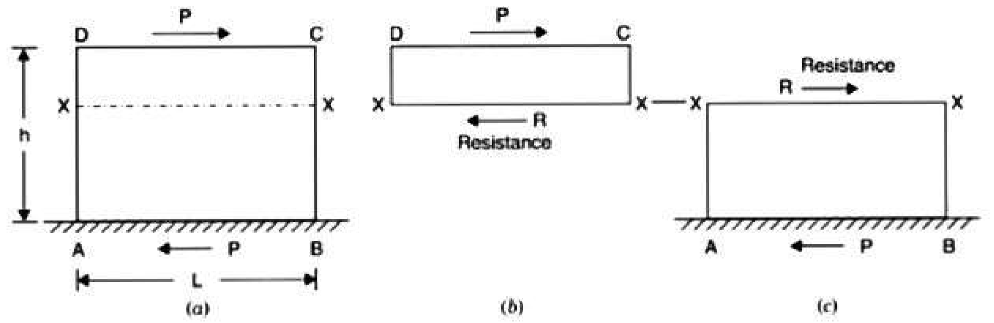


Fig. 1.4

Consider a section x-x (parallel to the applied force), which divides the block into two parts. The upper part will be in equilibrium if P = Resistance (R). This is shown in Fig. 1.4 (*b*). Similarly the lower part will be in equilibrium if P = Resistance (R) as shown in Fig. 1.4 (*c*). This resistance is known as shear resistance. And the shear resistance per unit area is known as shear stress which is represented by τ.

$$\therefore \text{ Shear stress, } \tau = \frac{\text{Shear resistance}}{\text{Shear area}} = \frac{R}{A}$$

$$= \frac{P}{L \times 1} \qquad (\because \quad R = P \text{ and } A = L \times 1) \quad ...(1.3)$$

Note that shear stress is tangential to the area over which it acts.

As the bottom face of the block is fixed, the face $ABCD$ will be distorted to ABC_1D_1 through an angle ϕ as a result of force P as shown in Fig. 1.4 (*d*).

And shear strain (ϕ) is given by,

$$\phi = \frac{\text{Transversal displacement}}{\text{Distance } AD}$$

or

$$\phi = \frac{DD_1}{AD} = \frac{dl}{h} \qquad ...(1.4)$$

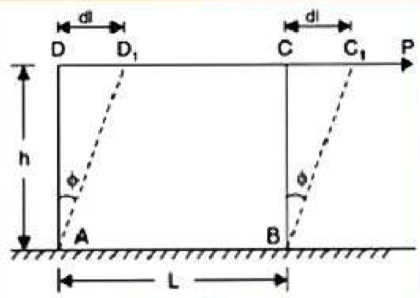


Fig. 1.4 (*d*)

Elasticity and Elastic Limit:-

When an external force acts on a body, the body tends to undergo some deformation. If the external force is removed and the body comes back to its original shape and size (which means the deformation disappears completely), the body is known as *elastic body*. This property, by virtue of which certain materials return back to their original position after the removal of the external force, is called *elasticity*.

The body will regain its previous shape and size only when the deformation caused by the external force, is within a certain limit. Thus there is a limiting value of force up to and within which, the deformation completely disappears on the removal of the force. The value of stress corresponding to this limiting force is known as the *elastic limit* of the material.

If the external force is so large that the stress exceeds the elastic limit, the material loses to some extent its property of elasticity. If now the force is removed, the material will not return to its original shape and size and there will be a residual deformation in the material.

Hooke's Law:-

Hooke's Law states that when a material is loaded within elastic limit, the stress is proportional to the strain produced by the stress. This means the ratio of the stress to the corresponding strain is a constant within the elastic limit. This constant is known as Modulus of Elasticity or Modulus of Rigidity or Elastic Modulii.

Mathematically, $\sigma \;\alpha\; e$

$$\sigma = E \times e$$

(Where E is Proportionality constant or it is also called as Young's Modulus of elasticity)

$$E = \frac{\sigma}{e}$$

Modulus of Elasticity or Young's modulus:-

The ratio of tensile stress or compressive stress to the corresponding strain is a constant. This ratio is known as Young's Modulus or Modulus of Elasticity and is denoted by *E*.

$$\therefore \qquad E = \frac{\text{Tensile stress}}{\text{Tensile strain}} \quad \text{or} \quad \frac{\text{Compressive stress}}{\text{Compressive strain}}$$

or

$$E = \frac{\sigma}{e} \qquad \text{...(1.5)}$$

1.7.1. Modulus of Rigidity or Shear Modulus. The ratio of shear stress to the corresponding shear strain within the elastic limit, is known as Modulus of Rigidity or Shear Modulus. This is denoted by *C* or *G* or *N*.

$$\therefore \qquad C \text{ (or } G \text{ or } N) = \frac{\text{Shear stress}}{\text{Shear strain}} = \frac{\tau}{\phi} \qquad \text{...(1.6)}$$

Let us define factor of safety also.

1.8. FACTOR OF SAFETY

It is defined as the ratio of ultimate tensile stress to the working (or permissible) stress. Mathematically it is written as

$$\text{Factor of safety} = \frac{\text{Ultimate stress}}{\text{Permissible stress}} \qquad \text{...(1.7)}$$

1.9. CONSTITUTIVE RELATIONSHIP BETWEEN STRESS AND STRAIN

1.9.1. For One-Dimensional Stress System. The relationship between stress and strain for a unidirectional stress (*i.e.*, for normal stress in one direction only) is given by **Hooke's law,** which states that when a material is loaded within its elastic limit, the normal stress developed is proportional to the strain produced. This means that the ratio of the normal stress to the corresponding strain is a constant within the elastic limit. This constant is represented by *E* and is known as modulus of elasticity or Young's modulus of elasticity.

$$\therefore \qquad \frac{\text{Normal stress}}{\text{Corresponding strain}} = \text{Constant} \quad \text{or} \quad \frac{\sigma}{e} = E$$

where σ = Normal stress, e = Strain and E = Young's modulus

or

$$e = \frac{\sigma}{E} \qquad \text{...[1.7 (A)]}$$

The above equation gives the stress and strain relation for the normal stress in one direction.

1.9.2. For Two-Dimensional Stress System. Before knowing the relationship between stress and strain for two-dimensional stress system, we shall have to define longitudinal strain, lateral strain, and Poisson's ratio.

1. Longitudinal strain. When a body is subjected to an axial tensile load, there is an increase in the length of the body. But at the same time there is a decrease in other dimensions of the body at right angles to the line of action of the applied load. Thus the body is having axial deformation and also deformation at right angles to the line of action of the applied load (*i.e.*, lateral deformation).

The ratio of axial deformation to the original length of the body is known as longitudinal (or linear) strain. The longitudinal strain is also defined as the deformation of the body per unit length in the direction of the applied load.

Let L = Length of the body,

P = Tensile force acting on the body,

δL = Increase in the length of the body in the direction of P.

Then, longitudinal strain $= \dfrac{\delta L}{L}$.

2. Lateral strain. The strain at right angles to the direction of applied load is known as lateral strain. Let a rectangular bar of length L, breadth b and depth d is subjected to an axial tensile load P as shown in Fig. 1.5. The length of the bar will increase while the breadth and depth will decrease.

Let δL = Increase in length,

δb = Decrease in breadth, and

δd = Decrease in depth.

$$\text{Then longitudinal strain} = \frac{\delta L}{L} \quad \text{...[1.7 }(B)]$$

$$\text{and} \quad \text{lateral strain} = \frac{\delta b}{b} \text{ or } \frac{\delta d}{d} \quad \text{...[1.7 }(C)]$$

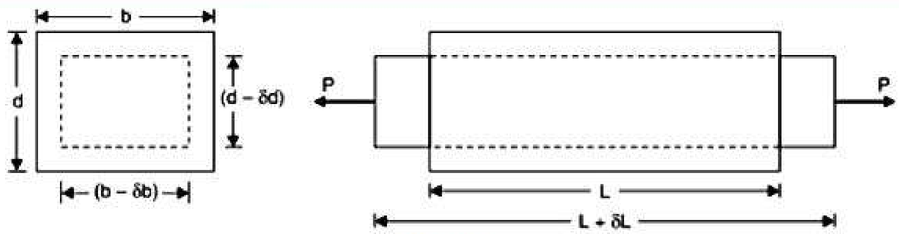


Fig. 1.5

Note. (*i*) If longitudinal strain is tensile, the lateral strains will be compressive.

(*ii*) If longitudinal strain is compressive then lateral strains will be tensile.

(*iii*) Hence every longitudinal strain in the direction of load is accompanied by lateral strains of the opposite kind in all directions perpendicular to the load.

3. Poisson's ratio. The ratio of lateral strain to the longitudinal strain is a constant for a given material, when the material is stressed within the elastic limit. This ratio is called **Poisson's ratio** and it is generally denoted by μ. Hence mathematically,

$$\text{Poisson's ratio, } \mu = \frac{\text{Lateral strain}}{\text{Longitudinal strain}} \quad \text{...[1.7 }(D)]$$

or Lateral strain $= \mu \times$ Longitudinal strain

As lateral strain is opposite in sign to longitudinal strain, hence algebraically, lateral strain is written as

$$\text{Lateral strain} = -\mu \times \text{Longitudinal strain} \quad \text{...[1.7 }(E)]$$

The value of Poisson's ratio varies from 0.25 to 0.33. For rubber, its value ranges from 0.45 to 0.50.

4. Relationship between stress and strain. Consider a two-dimensional figure *ABCD*, subjected to two mutually perpendicular stresses σ_1 and σ_2.

Refer to Fig. 1.5 (*a*).

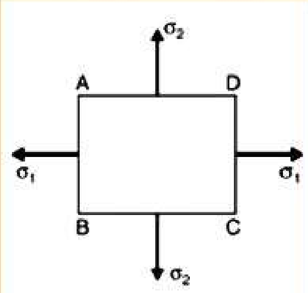


Fig. 1.5 (*a*)

Let σ_1 = Normal stress in *x*-direction

σ_2 = Normal stress in *y*-direction

Consider the strain produced by σ_1.

The stress σ_1 will produce strain in the direction of *x* and also in the direction of *y*. The strain in the direction of *x* will be longitudinal strain and will be equal to $\frac{\sigma_1}{E}$ whereas the strain in the direction of *y* will be lateral strain and will be equal to $-\mu \times \frac{\sigma_1}{E}$

($\because$ Lateral strain. = $-\mu \times$ longitudinal strain)

Now consider the strain produced by σ_2.

The stress σ_2 will produce strain in the direction of *y* and also in the direction of *x*. The strain in the direction of *y* will be longitudinal strain and will be equal to $\frac{\sigma_2}{E}$ whereas the strain in the direction of *x* will be lateral strain and will be equal to $-\mu \times \frac{\sigma_2}{E}$.

Let e_1 = Total strain in *x*-direction

e_2 = Total strain in *y*-direction

Now total strain in the direction of *x* due to stresses σ_1 and $\sigma_2 = \frac{\sigma_1}{E} - \mu \frac{\sigma_2}{E}$

Similarly total strain in the direction of *y* due to stresses σ_1 and $\sigma_2 = \frac{\sigma_2}{E} - \mu \frac{\sigma_1}{E}$

$$\therefore \quad e_1 = \frac{\sigma_1}{E} - \mu \frac{\sigma_2}{E} \quad \text{...[1.7 (F)]}$$

$$e_2 = \frac{\sigma_2}{E} - \mu \frac{\sigma_1}{E} \quad \text{...[1.7 (G)]}$$

The above two equations gives the stress and strain relationship for the two-dimensional stress system. In the above equations, tensile stress is taken to be positive whereas the compressive stress negative.

1.9.3. For Three-Dimensional Stress System. Fig. 1.5 (*b*) shows a three-dimensional body subjected to three orthogonal normal stresses σ_1, σ_2, σ_3 acting in the directions of *x*, *y* and *z* respectively.

Consider the strains produced by each stress separately.

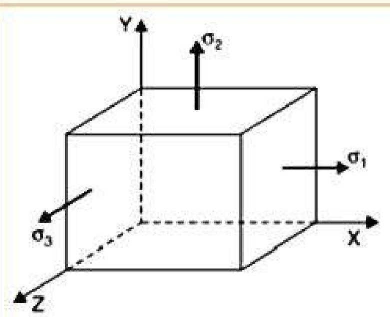


Fig. 1.5 (*b*)

The stress σ_1 will produce strain in the direction of *x* and also in the directions of *y* and *z*. The strain in the direction of *x* will be $\frac{\sigma_1}{E}$ whereas the strains in the direction of *y* and *z* will be $-\mu \frac{\sigma_1}{E}$.

Similarly the stress σ_2 will produce strain $\frac{\sigma_2}{E}$ in the direction of *y* and strain of $-\mu \frac{\sigma_2}{E}$ in the direction of *x* and *y* each.

Also the stress σ_3 will produce strain $\frac{\sigma_3}{E}$ in the direction of z and strain of $-\mu \times \frac{\sigma_3}{E}$ in the direction of x and y.

Total strain in the direction of x due to stresses σ_1, σ_2 and $\sigma_3 = \frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E} - \mu\frac{\sigma_3}{E}$.

Similarly total strains in the direction of y due to stresses σ_1, σ_2 and σ_3

$$= \frac{\sigma_2}{E} - \mu\frac{\sigma_3}{E} - \mu\frac{\sigma_1}{E}$$

and total strains in the direction of z due to stresses σ_1, σ_2 and σ_3

$$= \frac{\sigma_3}{E} - \mu\frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E}$$

Let e_1, e_2 and e_3 are total strains in the direction of x, y and z respectively. Then

$$e_1 = \frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E} - \mu\frac{\sigma_3}{E} \quad \text{...[1.7 (H)]}$$

$$e_2 = \frac{\sigma_2}{E} - \mu\frac{\sigma_3}{E} - \mu\frac{\sigma_1}{E} \quad \text{...[1.7 (I)]}$$

and

$$e_3 = \frac{\sigma_3}{E} - \mu\frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E} \quad \text{...[1.7 (J)]}$$

The above three equations give the stress and strain relationship for the three orthogonal normal stress system.

Problem 1.1. *A rod 150 cm long and of diameter 2.0 cm is subjected to an axial pull of 20 kN. If the modulus of elasticity of the material of the rod is 2×10^5 N/mm^2 ; determine :*

(i) the stress,

(ii) the strain, and

(iii) the elongation of the rod.

Sol. Given : Length of the rod, $L = 150$ cm

Diameter of the rod, $D = 2.0$ cm $= 20$ mm

∴ Area, $A = \frac{\pi}{4}(20)^2 = 100\pi$ mm^2

Axial pull, $P = 20$ kN $= 20{,}000$ N

Modulus of elasticity, $E = 2.0 \times 10^5$ N/mm^2

(*i*) The stress (σ) is given by equation (1.1) as

$$\sigma = \frac{P}{A} = \frac{20000}{100\pi} = \mathbf{63.662\ N/mm^2}. \quad \textbf{Ans.}$$

(*ii*) Using equation (1.5), the strain is obtained as

$$E = \frac{\sigma}{e}.$$

∴ Strain, $e = \frac{\sigma}{E} = \frac{63.662}{2 \times 10^5} = \mathbf{0.000318}.$ **Ans.**

(*iii*) Elongation is obtained by using equation (1.2) as

$$e = \frac{dL}{L}.$$

∴ Elongation, $dL = e \times L$

$$= 0.000318 \times 1500$$

$$= 0.477 \text{ mm}$$

2.4. POISSON'S RATIO

The ratio of lateral strain to the longitudinal strain is a constant for a given material, when the material is stressed within the elastic limit. This ratio is called **Poisson's ratio** and it is generally denoted by μ. Hence mathematically,

$$\text{Poisson's ratio, } \mu = \frac{\text{Lateral strain}}{\text{Longitudinal strain}} \quad ...(2.3)$$

or

$$\text{Lateral strain} = \mu \times \text{longitudinal strain}$$

As lateral strain is opposite in sign to longitudinal strain, hence algebraically, the lateral strain is written as

$$\text{Lateral strain} = -\mu \times \text{longitudinal strain} \quad ...[2.3\ (A)]$$

The value of Poisson's ratio varies from 0.25 to 0.33. For rubber, its value ranges from 0.45 to 0.50.

2.5. VOLUMETRIC STRAIN

The ratio of change in volume to the original volume of a body (when the body is subjected to a single force or a system of forces) is called volumetric strain. It is denoted by e_v.

Mathematically, volumetric strain is given by

$$e_v = \frac{\delta V}{V}$$

where δV = Change in volume, and

V = Original volume.

2.7. BULK MODULUS

When a body is subjected to the mutually perpendicular like and equal direct stresses, the ratio of direct stress to the corresponding volumetric strain is found to be constant for a given material when the deformation is within a certain limit. This ratio is known as bulk modulus and is usually denoted by K. Mathematically bulk modulus is given by

$$K = \frac{\text{Direct stress}}{\text{Volumetric strain}} = \frac{\sigma}{\left(\frac{dV}{V}\right)} \quad ...(2.9)$$

Problem 1.2. *Find the minimum diameter of a steel wire, which is used to raise a load of 4000 N if the stress in the rod is not to exceed 95 MN/m².*

Sol. Given : Load, $P = 4000$ N

Stress, $\sigma = 95\ \text{MN/m}^2 = 95 \times 10^6\ \text{N/m}^2$ $\quad (\because \text{M} = \text{Mega} = 10^6)$

$= 95\ \text{N/mm}^2$ $\quad (\because 10^6\ \text{N/m}^2 = 1\ \text{N/mm}^2)$

Let D = Diameter of wire in mm

$\therefore$ Area, $A = \frac{\pi}{4} D^2$

Now stress $= \frac{\text{Load}}{\text{Area}} = \frac{P}{A}$

$$95 = \frac{4000}{\frac{\pi}{4} D^2} = \frac{4000 \times 4}{\pi D^2} \quad \text{or} \quad D^2 = \frac{4000 \times 4}{\pi \times 95} = 53.61$$

$\therefore$ $D = \mathbf{7.32\ mm}$. **Ans.**

Problem 2.1. *Determine the changes in length, breadth and thickness of a steel bar which is 4 m long, 30 mm wide and 20 mm thick and is subjected to an axial pull of 30 kN in the direction of its length. Take $E = 2 \times 10^5$ N/mm^2 and Poisson's ratio = 0.3.*

Sol. Given :

Length of the bar,	$L = 4$ m $= 4000$ mm
Breadth of the bar,	$b = 30$ mm
Thickness of the bar,	$t = 20$ mm
∴ Area of cross-section,	$A = b \times t = 30 \times 20 = 600$ mm^2
Axial pull,	$P = 30$ kN $= 30000$ N
Young's modulus,	$E = 2 \times 10^5$ N/mm^2
Poisson's ratio,	$\mu = 0.3.$

Now strain in the direction of load (or longitudinal strain),

$$= \frac{\text{Stress}}{E} = \frac{\text{Load}}{\text{Area} \times E} \qquad \left(\because \text{ Stress} = \frac{\text{Load}}{\text{Area}}\right)$$

$$= \frac{P}{A.E.} = \frac{30000}{600 \times 2 \times 10^5} = 0.00025.$$

But longitudinal strain $= \frac{\delta L}{L}$.

$$\therefore \qquad \frac{\delta L}{L} = 0.00025.$$

∴ δL (or change in length) $= 0.00025 \times L$

$= 0.00025 \times 4000 =$ **1.0 mm. Ans.**

Using equation (2.3),

$$\text{Poisson's ratio} = \frac{\text{Lateral strain}}{\text{Longitudinal strain}}$$

or
$$0.3 = \frac{\text{Lateral strain}}{0.00025}$$

∴ Lateral strain $= 0.3 \times 0.00025 = 0.000075.$

We know that

$$\text{Lateral strain} = \frac{\delta b}{b} \text{ or } \frac{\delta d}{d}\left(\text{or } \frac{\delta t}{t}\right)$$

∴ $\delta b = b \times$ Lateral strain

$= 30 \times 0.000075 =$ **0.00225 mm. Ans.**

Similarly, $\delta t = t \times$ Lateral strain

$= 20 \times 0.000075 =$ **0.0015 mm. Ans.**

Problem 2.2. *Determine the value of Young's modulus and Poisson's ratio of a metallic bar of length 30 cm, breadth 4 cm and depth 4 cm when the bar is subjected to an axial compressive load of 400 kN. The decrease in length is given as 0.075 cm and increase in breadth is 0.003 cm.*

Sol. Given :

Length, $L = 30$ cm ; Breadth, $b = 4$ cm ; and Depth, $d = 4$ cm.

∴ Area of cross-section, $A = b \times d = 4 \times 4$

$= 16$ cm$^2 = 16 \times 100 = 1600$ mm^2

Axial compressive load,	P	$= 400$ kN $= 400 \times 1000$ N
Decrease in length,	δL	$= 0.075$ cm
Increase in breadth,	δb	$= 0.003$ cm

$$\text{Longitudinal strain} = \frac{\delta L}{L} = \frac{0.075}{30} = 0.0025$$

$$\text{Lateral strain} = \frac{\delta b}{b} = \frac{0.003}{4} = 0.00075.$$

$$\text{Poisson's ratio} = \frac{\text{Lateral strain}}{\text{Longitudinal strain}} = \frac{0.00075}{0.0025} = \mathbf{0.3.} \quad \text{Ans.}$$

$$\text{Longitudinal strain} = \frac{\text{Stress}}{E} = \frac{P}{A \times E} \qquad \left(\because \text{Stress} = \frac{\text{Load}}{\text{Area}} = \frac{P}{A}\right)$$

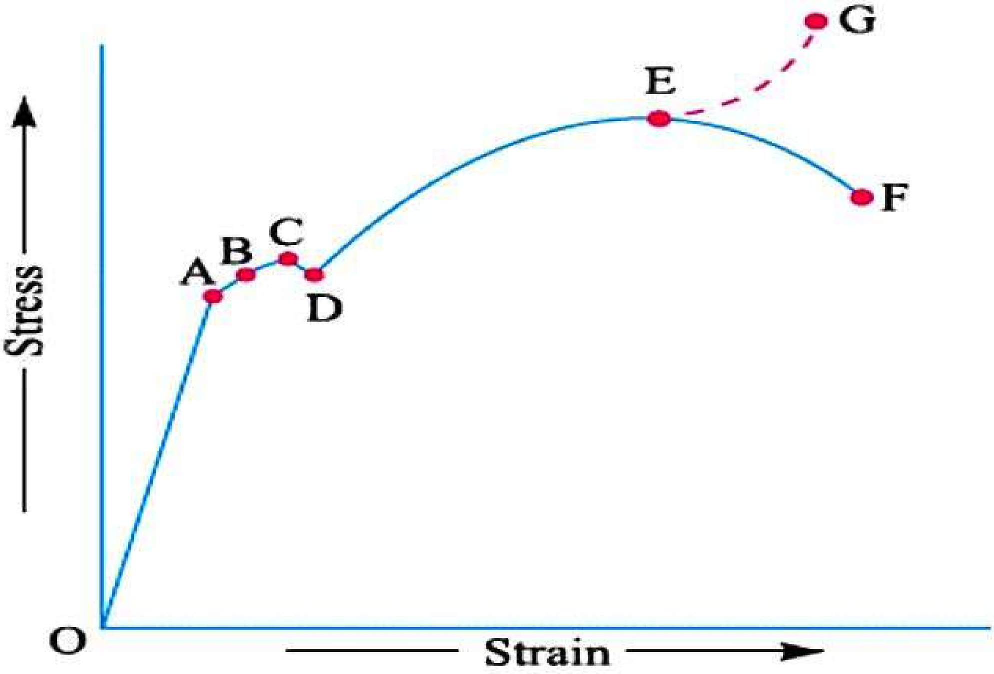
$$0.0025 = \frac{400000}{1600 \times E}$$

$$\therefore \quad E = \frac{400000}{1600 \times 0.0025} = \mathbf{1 \times 10^5\ N/mm^2.} \quad \text{Ans.}$$

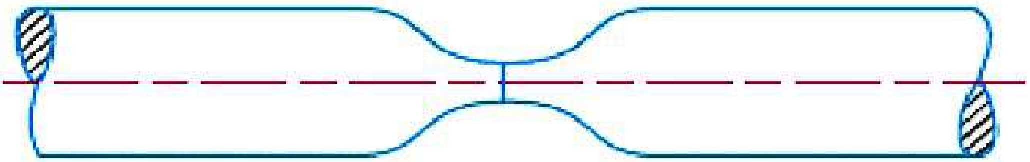
Stress-strain Diagram for Ductile Material (Mild Steel)

In designing various parts of a machine, it is necessary to know how the material will function in service. For this, certain characteristics or properties of the material should be known. The mechanical properties mostly used in mechanical engineering practice are commonly determined from a standard tensile test. This test consists of gradually loading a standard specimen of a material and noting the corresponding values of load and elongation until the specimen fractures. The load is applied and measured by a testing machine. The stress is determined by dividing the load values by the original cross-sectional area of the specimen. The elongation is measured by determining the amounts that two reference points on the specimen are moved apart by the action of the machine. The original distance between the two reference points is known as **gauge length.** The strain is determined by dividing the elongation values by the gauge length.

The values of the stress and corresponding strain are used to draw the stress-strain diagram of the material tested. A stress-strain diagram for a mild steel under tensile test is shown in fig given below. The various properties of the material are discussed below:



Stress-Strain diagram for a mild steel specimen (fig. a)


Shape of the Specimen after elongation (fig. b)

1. Proportional limit. We see from the diagram that from point O to A is a straight line, which represents that the stress is proportional to strain. Beyond point A, the curve slightly deviates from the straight line. It is thus obvious, that Hooke's law holds good up to point A and it is known as **proportional limit**. It is defined as that stress at which the stress-strain curve begins to deviate from the straight line.

2. Elastic limit. It may be noted that even if the load is increased beyond point A upto the point B, the material will regain its shape and size when the load is removed. This means that the material has elastic properties up to the point B. This point is known as **elastic limit.** It is defined as the stress developed in the material without any permanent set.

Note: Since the above two limits are very close to each other, therefore, for all practical purposes these are taken to be equal.

3. Yield point. If the material is stressed beyond point B, the plastic stage will reach i.e. on the removal of the load, the material will not be able to recover its original size and shape. A little consideration will show that beyond point B, the strain increases at a faster rate with any increase in the stress until the point C is reached. At this point, the material yields before the load and there is an appreciable strain without any increase in stress. In case of mild steel, it will be seen that a small load drops to D, immediately after yielding commences. Hence there are two yield points C and D. The points C and D are called the **upper** and **lower yield points** respectively. The stress corresponding to yield point is known as **yield point stress.**

4. Ultimate stress. At D, the specimen regains some strength and higher values of stresses are required for higher strains, than those between A and D. The stress (or load) goes on increasing till the point E is reached. The gradual increase in the strain (or length) of the specimen is followed with the uniform reduction of its cross-sectional area. The work done, during stretching the specimen, is transformed largely into heat and the specimen becomes hot. At E, the stress, which attains its maximum value is known as **ultimate stress.** It is defined as the largest stress obtained by dividing the largest value of the load reached in a test to the original cross-sectional area of the test piece.

5. Breaking stress. After the specimen has reached the ultimate stress, a neck is formed, which decreases the cross-sectional area of the specimen, as shown in Fig. (b). A little consideration will show that the stress (or load) necessary to break away the specimen, is less than the maximum stress. The stress is, therefore, reduced until the specimen breaks away at point F. The stress corresponding to point F is known as **breaking stress.**

Note: The breaking stress (i.e. stress at F which is less than at E) appears to be somewhat misleading. As the formation of a neck takes place at E which reduces the cross-sectional area, it causes the specimen suddenly to fail at F. If for each value of the strain between E and F, the tensile load is divided by the reduced cross-sectional area at the narrowest part of the neck, then the true stress-strain curve will follow the dotted line EG.
However, it is an established practice, to calculate strains on the basis of original cross-sectional area of the specimen.

6. Percentage reduction in area. It is the difference between the original cross-sectional area and cross-sectional area at the neck (i.e. where the fracture takes place). This difference is expressed as percentage of the original cross-sectional area.

Let A = Original cross-sectional area, and

a = Cross-sectional area at the neck.

Then reduction in area = $A - a$

and percentage reduction in area = $\frac{A-a}{A} \times 100$

7. Percentage elongation. It is the percentage increase in the standard gauge length (i.e. original length) obtained by measuring the fractured specimen after bringing the broken parts together.

Let l = Gauge length or original length, and

L = Length of specimen after fracture or final length.

∴ Elongation = $L - l$

And percentage elongation = $\frac{L-l}{l} \times 100$

Problem 1.3. *Find the Young's Modulus of a brass rod of diameter 25 mm and of length 250 mm which is subjected to a tensile load of 50 kN when the extension of the rod is equal to 0.3 mm.*

Sol. Given : Dia. of rod, $D = 25$ mm

$\therefore$ Area of rod, $A = \frac{\pi}{4}(25)^2 = 490.87 \text{ mm}^2$

Tensile load, $P = 50 \text{ kN} = 50 \times 1000 = 50{,}000 \text{ N}$

Extension of rod, $dL = 0.3$ mm

Length of rod, $L = 250$ mm

Stress (σ) is given by equation (1.1), as

$$\sigma = \frac{P}{A} = \frac{50{,}000}{490.87} = 101.86 \text{ N/mm}^2.$$

Strain (e) is given by equation (1.2), as

$$e = \frac{dL}{L} = \frac{0.3}{250} = 0.0012.$$

Using equation (1.5), the Young's Modulus (E) is obtained, as

$$E = \frac{\text{Stress}}{\text{Strain}} = \frac{101.86 \text{ N/mm}^2}{0.0012} = 84883.33 \text{ N/mm}^2$$

$= \mathbf{84883.33 \times 10^6\ N/m^2}$. **Ans.** $(\because\ 1 \text{ N/mm}^2 = 10^6 \text{ N/m}^2)$

$= 84.883 \times 10^9 \text{ N/m}^2 = \mathbf{84.883\ GN/m^2}$. **Ans.** $(\because\ 10^9 = \text{G})$

Problem 1.4. *A tensile test was conducted on a mild steel bar. The following data was obtained from the test :*

(i) Diameter of the steel bar	*= 3 cm*
(ii) Gauge length of the bar	*= 20 cm*
(iii) Load at elastic limit	*= 250 kN*
(iv) Extension at a load of 150 kN	*= 0.21 mm*
(v) Maximum load	*= 380 kN*
(vi) Total extension	*= 60 mm*
(vii) Diameter of the rod at the failure	*= 2.25 cm.*

Determine : (a) the Young's modulus, (b) the stress at elastic limit, (c) the percentage elongation, and (d) the percentage decrease in area.

Sol. Area of the rod, $A = \frac{\pi}{4} D^2 = \frac{\pi}{4}(3)^2 \text{ cm}^2$

$= 7.0685 \text{ cm}^2 = 7.0685 \times 10^{-4} \text{ m}^2.$ $\left[\because\ \text{cm}^2 = \left(\frac{1}{100}\text{ m}\right)^2\right]$

(*a*) To find Young's modulus, first calculate the value of stress and strain within elastic limit. The load at elastic limit is given but the extension corresponding to the load at elastic limit is not given. But a load of 150 kN (which is within elastic limit) and corresponding extension of 0.21 mm are given. Hence these values are used for stress and strain within elastic limit

$\therefore$ Stress $= \frac{\text{Load}}{\text{Area}} = \frac{150 \times 1000}{7.0685 \times 10^{-4}} \text{ N/m}^2$ $(\because\ 1 \text{ kN} = 1000 \text{ N})$

$= 21220.9 \times 10^4 \text{ N/m}^2$

and Strain $= \frac{\text{Increase in length (or Extension)}}{\text{Original length (or Gauge length)}}$

$= \frac{0.21 \text{ mm}}{20 \times 10 \text{ mm}} = 0.00105$

$\therefore$ Young's Modulus,

$$E = \frac{\text{Stress}}{\text{Strain}} = \frac{21220.9 \times 10^4}{0.00105} = 20209523 \times 10^4 \text{ N/m}^2$$

$$= 202.095 \times 10^9 \text{ N/m}^2 \qquad (\because \ 10^9 = \text{Giga} = \text{G})$$

$$= \mathbf{202.095 \ GN/m^2}.$$ **Ans.**

(*b*) The stress at the elastic limit is given by,

$$\text{Stress} = \frac{\text{Load at elastic limit}}{\text{Area}} = \frac{250 \times 1000}{7.0685 \times 10^{-4}}$$

$$= 35368 \times 10^4 \text{ N/m}^2$$

$$= 353.68 \times 10^6 \text{ N/m}^2 \qquad (\because \ 10^6 = \text{Mega} = \text{M})$$

$$= \mathbf{353.68 \ MN/m^2}.$$ **Ans.**

(*c*) The percentage elongation is obtained as,

Percentage elongation

$$= \frac{\text{Total increase in length}}{\text{Original length (or Gauge length)}} \times 100$$

$$= \frac{60 \text{ mm}}{20 \times 10 \text{ mm}} \times 100 = \mathbf{30\%}.$$ **Ans.**

(*d*) The percentage decrease in area is obtained as,

Percentage decrease in area

$$= \frac{\text{(Original area − Area at the failure)}}{\text{Original area}} \times 100$$

$$= \frac{\left(\frac{\pi}{4} \times 3^2 - \frac{\pi}{4} \times 2.25^2\right)}{\frac{\pi}{4} \times 3^2} \times 100$$

$$= \left(\frac{3^2 - 2.25^2}{3^2}\right) \times 100 = \frac{(9 - 5.0625)}{9} \times 100 = \mathbf{43.75\%}.$$ **Ans.**

Problem 1.6. ***The ultimate stress, for a hollow steel column which carries an axial load of 1.9 MN is 480 N/mm². If the external diameter of the column is 200 mm, determine the internal diameter. Take the factor of safety as 4.***

Sol. Given :

Ultimate stress $= 480 \text{ N/mm}^2$

Axial load, $P = 1.9 \text{ MN} = 1.9 \times 10^6 \text{ N}$ $(\because \text{M} = 10^6)$

$= 1900000 \text{ N}$

External dia., $D = 200$ mm

Factor of safety $= 4$

Let d = Internal diameter in mm

$\therefore$ Area of cross-section of the column,

$$A = \frac{\pi}{4}(D^2 - d^2) = \frac{\pi}{4}(200^2 - d^2) \text{ mm}^2$$

Using equation (1.7), we get

$$\text{Factor of safety} = \frac{\text{Ultimate stress}}{\text{Working stress or Permissible stress}}$$

$$\therefore \quad 4 = \frac{480}{\text{Working stress}}$$

or Working stress $= \dfrac{480}{4} = 120 \text{ N/mm}^2$

$\therefore$ $\sigma = 120 \text{ N/mm}^2$

Now using equation (1.1), we get

$$\sigma = \frac{P}{A} \quad \text{or} \quad 120 = \frac{1900000}{\frac{\pi}{4}(200^2 - d^2)} = \frac{1900000 \times 4}{\pi(40000 - d^2)}$$

or

$$40000 - d^2 = \frac{1900000 \times 4}{\pi \times 120} = 20159.6$$

$$d^2 = 40000 - 20159.6 = 19840.4$$

$$d = \mathbf{140.85 \text{ mm.}}$$ **Ans.**

ANALYSIS OF BARS OF VARYING SECTIONS

A bar of different lengths and of different diameters (and hence of different cross-sectional areas) is shown in Fig. 1.6 (*a*). Let this bar is subjected to an axial load P.

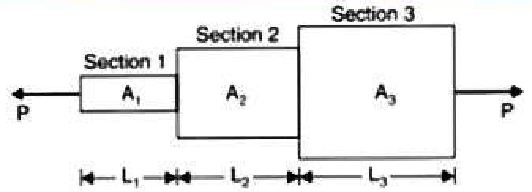


Fig. 1.6 (*a*)

Though each section is subjected to the same axial load P, yet the stresses, strains and change in lengths will be different. The total change in length will be obtained by adding the changes in length of individual section.

Let P = Axial load acting on the bar,

L_1 = Length of section 1,

A_1 = Cross-sectional area of section 1,

L_2, A_2 = Length and cross-sectional area of section 2,

L_3, A_3 = Length and cross-sectional area of section 3, and

E = Young's modulus for the bar.

Then stress for the section 1,

$$\sigma_1 = \frac{\text{Load}}{\text{Area of section 1}} = \frac{P}{A_1}.$$

Similarly stresses for the section 2 and section 3 are given as,

$$\sigma_2 = \frac{P}{A_2} \quad \text{and} \quad \sigma_3 = \frac{P}{A_3}$$

Using equation (1.5), the strains in different sections are obtained.

$\therefore$ Strain of section 1, $e_1 = \frac{\sigma_1}{E} = \frac{P}{A_1 E}$ $\quad\left(\because \sigma_1 = \frac{P}{A_1}\right)$

Similarly the strains of section 2 and of section 3 are,

$$e_2 = \frac{\sigma_2}{E} = \frac{P}{A_2 E} \quad \text{and} \quad e_3 = \frac{\sigma_3}{E} = \frac{P}{A_3 E}.$$

But strain in section 1 $= \frac{\text{Change in length of section 1}}{\text{Length of section 1}}$

or $\quad e_1 = \frac{dL_1}{L_1}$

where dL_1 = change in length of section 1.

$\therefore$ Change in length of section 1, $dL_1 = e_1 L_1$

$$= \frac{PL_1}{A_1 E} \quad \left(\because e_1 = \frac{P}{A_1 E}\right)$$

Similarly changes in length of section 2 and of section 3 are obtained as :

Change in length of section 2, $dL_2 = e_2 L_2$

$$= \frac{PL_2}{A_2 E} \quad \left(\because e_2 = \frac{P}{A_2 E}\right)$$

and change in length of section 3, $dL_3 = e_3 L_3$

$$= \frac{PL_3}{A_3 E} \quad \left(\because e_3 = \frac{P}{A_3 E}\right)$$

∴ Total change in the length of the bar,

$$dL = dL_1 + dL_2 + dL_3 = \frac{PL_1}{A_1E} + \frac{PL_2}{A_2E} + \frac{PL_3}{A_3E}$$

$$= \frac{P}{E}\left[\frac{L_1}{A_1} + \frac{L_2}{A_2} + \frac{L_3}{A_3}\right] \quad ...(1.8)$$

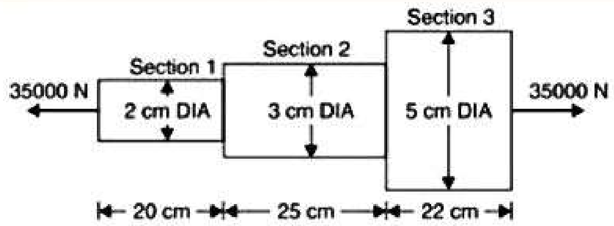
Equation (1.8) is used when the Young's modulus of different sections is same. If the Young's modulus of different sections is different, then total change in length of the bar is given by,

$$dL = P\left[\frac{L_1}{E_1A_1} + \frac{L_2}{E_2A_2} + \frac{L_3}{E_3A_3}\right] \quad ...(1.9)$$

Problem 1.8. *An axial pull of 35000 N is acting on a bar consisting of three lengths as shown in Fig. 1.6 (b). If the Young's modulus = 2.1 × 10⁵ N/mm², determine :*

(i) stresses in each section and

(ii) total extension of the bar.



Sol. Given :

Axial pull, $P = 35000$ N

Length of section 1, $L_1 = 20$ cm $= 200$ mm

Dia. of section 1, $D_1 = 2$ cm $= 20$ mm

∴ Area of section 1, $A_1 = \frac{\pi}{4}(20^2) = 100\pi$ mm²

Length of section 2, $L_2 = 25$ cm $= 250$ mm

Dia. of section 2, $D_2 = 3$ cm $= 30$ mm

∴ Area of section 2, $A_2 = \frac{\pi}{4}(30^2) = 225\pi$ mm²

Length of section 3, $L_3 = 22$ cm $= 220$ mm

Dia. of section 3, $D_3 = 5$ cm $= 50$ mm

∴ Area of section 3, $A_3 = \frac{\pi}{4}(50^2) = 625\pi$ mm²

Young's modulus, $E = 2.1 \times 10^5$ N/mm².

(i) Stresses in each section

Stress in section 1, $\sigma_1 = \frac{\text{Axial load}}{\text{Area of section 1}}$

$$= \frac{P}{A_1} = \frac{35000}{100\pi} = \mathbf{111.408\ N/mm^2}.$$ **Ans.**

Stress in section 2, $\sigma_2 = \frac{P}{A_2} = \frac{35000}{225 \times \pi} = \mathbf{49.5146\ N/mm^2}$. **Ans.**

Stress in section 3, $\sigma_3 = \frac{P}{A_3} = \frac{35000}{625 \times \pi} = \mathbf{17.825\ N/mm^2}$. **Ans.**

(ii) Total extension of the bar

Using equation (1.8), we get

Total extension $= \frac{P}{E}\left(\frac{L_1}{A_1} + \frac{L_2}{A_2} + \frac{L_3}{A_3}\right)$

$$= \frac{35000}{2.1 \times 10^5}\left(\frac{200}{100\,\pi} + \frac{250}{225 \times \pi} + \frac{220}{625 \times \pi}\right)$$

$$= \frac{35000}{2.1 \times 10^5}\,(6.366 + 3.536 + 1.120) = \mathbf{0.183\ mm.}\quad \textbf{Ans.}$$

1.10.1. Principle of Superposition. When a number of loads are acting on a body, the resulting strain, according to principle of superposition, will be the algebraic sum of strains caused by individual loads.

While using this principle for an elastic body which is subjected to a number of direct forces (tensile or compressive) at different sections along the length of the body, first the free body diagram of individual section is drawn. Then the deformation of the each section is obtained. The total deformation of the body will be then equal to the algebraic sum of deformations of the individual sections.

Problem 1.12. *A member ABCD is subjected to point loads P_1, P_2, P_3 and P_4 as shown in Fig. 1.11.*

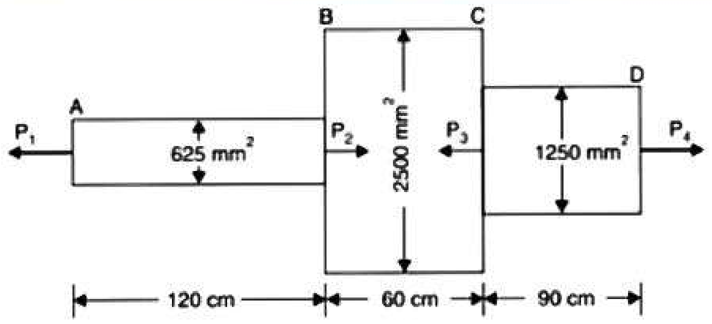


Fig. 1.11

Calculate the force P_2 necessary for equilibrium, if P_1 = 45 kN, P_3 = 450 kN and P_4 = 130 kN. Determine the total elongation of the member, assuming the modulus of elasticity to be 2.1×10^5 N/mm².

Sol. Given :

Part *AB* :	Area,	A_1 = 625 mm² and
	Length,	L_1 = 120 cm = 1200 mm
Part *BC* :	Area,	A_2 = 2500 mm² and
	Length,	L_2 = 60 cm = 600 mm
Part *CD* :	Area,	A_3 = 1250 mm² and
	Length,	L_3 = 90 cm = 900 mm
Value of		E = 2.1×10^5 N/mm².

Value of P_2 necessary for equilibrium

Resolving the forces on the rod along its axis (*i.e.*, equating the forces acting towards right to those acting towards left), we get

$$P_1 + P_3 = P_2 + P_4$$

But $P_1 = 45$ kN,

$P_3 = 450$ kN and $P_4 = 130$ kN

$\therefore$ $45 + 450 = P_2 + 130$ or $P_2 = 495 - 130 = 365$ kN

The force of 365 kN acting at *B* is split into two forces of 45 kN and 320 kN (*i.e.*, 365 – 45 = 320 kN).

The force of 450 kN acting at *C* is split into two forces of 320 kN and 130 kN (*i.e.*, 450 – 320 = 130 kN) as shown in Fig. 1.12.

From Fig. 1.12, it is clear that part *AB* is subjected to a tensile load of 45 kN, part *BC* is subjected to a compressive load of 320 kN and part *CD* is subjected to a tensile load 130 kN.

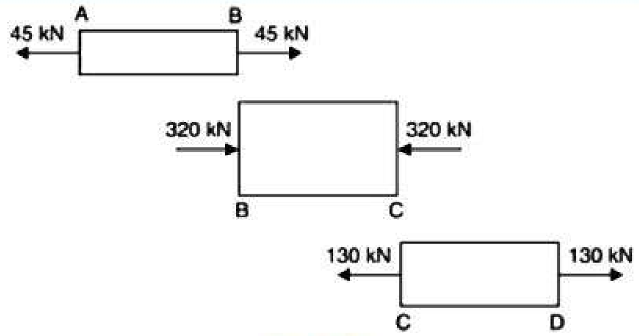


Fig. 1.12

Hence for part *AB*, there will be increase in length ; for part *BC* there will be decrease in length and for part *CD* there will be increase in length.

$\therefore$ Increase in length of *AB*

$$= \frac{P}{A_1 E} \times L_1 = \frac{45000}{625 \times 2.1 \times 10^5} \times 1200 \qquad (\because P = 45 \text{ kN} = 45000 \text{ N})$$

$$= 0.4114 \text{ mm}$$

Decrease in length of *BC*

$$= \frac{P}{A_2 E} \times L_2 = \frac{320,000}{2500 \times 2.1 \times 10^5} \times 600 \qquad (\because P = 320 \text{ kN} = 320000)$$

$$= 0.3657 \text{ mm}$$

Increase in length of *CD*

$$= \frac{P}{A_3 E} \times L_3 = \frac{130,000}{1250 \times 2.1 \times 10^5} \times 900 \qquad (\because P = 130 \text{ kN} = 130000)$$

$$= 0.4457 \text{ mm}$$

Total change in the length of member

$$= 0.4114 - 0.3657 + 0.4457$$

(Taking +ve sign for increase in length and –ve sign for decrease in length)

= **0.4914 mm (extension). Ans.**

1.13. ANALYSIS OF BARS OF COMPOSITE SECTIONS

A bar, made up of two or more bars of equal lengths but of different materials rigidly fixed with each other and behaving as one unit for extension or compression when subjected to an axial tensile or compressive loads, is called a composite bar. For the composite bar the following two points are important :

1. The extension or compression in each bar is equal. Hence deformation per unit length *i.e.*, strain in each bar is equal.

2. The total external load on the composite bar is equal to the sum of the loads carried by each different material.

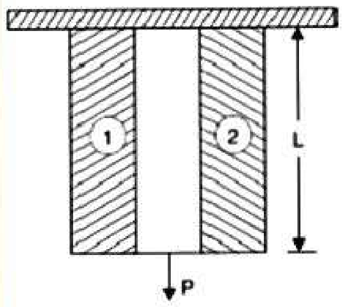


Fig. 1.15

Fig. 1.15 shows a composite bar made up of two different materials.

Let P = Total load on the composite bar,

L = Length of composite bar and also length of bars of different materials,

A_1 = Area of cross-section of bar 1,

A_2 = Area of cross-section of bar 2,

E_1 = Young's Modulus of bar 1,

E_2 = Young's Modulus of bar 2,

P_1 = Load shared by bar 1,

P_2 = Load shared by bar 2,

σ_1 = Stress induced in bar 1, and

σ_2 = Stress induced in bar 2.

Now the total load on the composite bar is equal to the sum of the load carried by the two bars.

$$\therefore \quad P = P_1 + P_2 \quad \text{...}(i)$$

The stress in bar 1, $= \dfrac{\text{Load carried by bar 1}}{\text{Area of cross-section of bar 1}}$.

$$\therefore \quad \sigma_1 = \frac{P_1}{A_1} \quad \text{or} \quad P_1 = \sigma_1 A_1 \quad \text{...}(ii)$$

Similarly stress in bar 2, $$\sigma_2 = \frac{P_2}{A_2} \quad \text{or} \quad P_2 = \sigma_2 A_2 \quad \text{...}(iii)$$

Substituting the values of P_1 and P_2 in equation (*i*), we get

$$P = \sigma_1 A_1 + \sigma_2 A_2 \quad \text{...}(iv)$$

Since the ends of the two bars are rigidly connected, each bar will change in length by the same amount. Also the length of each bar is same and hence the ratio of change in length to the original length (*i.e.*, strain) will be same for each bar.

But strain in bar 1, $= \dfrac{\text{Stress in bar 1}}{\text{Young's modulus of bar 1}} = \dfrac{\sigma_1}{E_1}$.

Similarly strain in bar 2, $= \dfrac{\sigma_2}{E_2}$.

But strain in bar 1 = Strain in bar 2

$$\therefore \quad = \frac{\sigma_1}{E_1} = \frac{\sigma_2}{E_2} \quad \text{...}(v)$$

From equations (*iv*) and (*v*), the stresses σ_1 and σ_2 can be determined. By substituting the values of σ_1 and σ_2 in equations (*ii*) and (*iii*), the load carried by different materials may be computed.

Modular Ratio. The ratio of $\frac{E_1}{E_2}$ is called the modular ratio of the first material to the second.

Problem 1.19. *A steel rod of 3 cm diameter is enclosed centrally in a hollow copper tube of external diameter 5 cm and internal diameter of 4 cm. The composite bar is then subjected to an axial pull of 45000 N. If the length of each bar is equal to 15 cm, determine :*

(i) The stresses in the rod and tube, and

(ii) Load carried by each bar.

Take E for steel = 2.1×10^5 N/mm² and for copper = 1.1×10^5 N/mm².

Sol. Given :

Dia. of steel rod = 3 cm = 30 mm

∴ Area of steel rod,

$$A_s = \frac{\pi}{4}(30)^2 = 706.86 \text{ mm}^2$$

External dia. of copper tube

= 5 cm = 50 mm

Internal dia. of copper tube

= 4 cm = 40 mm

∴ Area of copper tube,

$$A_c = \frac{\pi}{4}[50^2 - 40^2] \text{ mm}^2 = 706.86 \text{ mm}^2$$

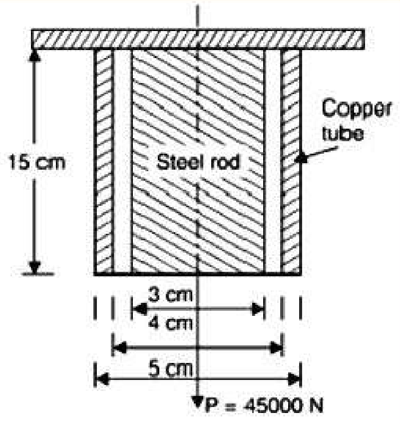


Fig. 1.16

Axial pull on composite bar, $P = 45000$ N

Length of each bar, $L = 15$ cm

Young's modulus for steel, $E_s = 2.1 \times 10^5$ N/mm²

Young's modulus for copper, $E_c = 1.1 \times 10^5$ N/mm²

(i) The stress in the rod and tube

Let σ_s = Stress in steel,

P_s = Load carried by steel rod,

σ_c = Stress in copper, and

P_c = Load carried by copper tube.

Now strain in steel = Strain in copper

or $$\frac{\sigma_s}{E_s} = \frac{\sigma_c}{E_c} \qquad \left(\because \frac{\sigma}{E} = \text{strain}\right)$$

∴ $$\sigma_s = \frac{E_s}{E_c} \times \sigma_c = \frac{2.1 \times 10^5}{1.1 \times 10^5} \times \sigma_c = 1.909\ \sigma_c \qquad ...(i)$$

Now $$\text{stress} = \frac{\text{Load}}{\text{Area}}, \quad \therefore \quad \text{Load} = \text{Stress} \times \text{Area}$$

Load on steel + Load on copper = Total load

$$\sigma_s \times A_s + \sigma_c \times A_c = P \qquad (\because \text{Total load} = P)$$

or $$1.909\ \sigma_c \times 706.86 + \sigma_c \times 706.86 = 45000$$

or $$\sigma_c (1.909 \times 706.86 + 706.86) = 45000$$

or $$2056.25\ \sigma_c = 45000$$

∴ $$\sigma_c = \frac{45000}{2056.25} = \textbf{21.88 N/mm}^2. \quad \textbf{Ans.}$$

Substituting the value of σ_c in equation (*i*), we get

$$\sigma_s = 1.909 \times 21.88 \text{ N/mm}^2$$

$$= \textbf{41.77 N/mm}^2. \quad \textbf{Ans.}$$

(ii) Load carried by each bar

As load = Stress × Area

∴ Load carried by steel rod,

$$P_s = \sigma_s \times A_s$$
$$= 41.77 \times 706.86 = \mathbf{29525.5\ N.}\ \textbf{Ans.}$$

Load carried by copper tube,

$$P_c = 45000 - 29525.5$$
$$= \mathbf{15474.5\ N.}\ \textbf{Ans.}$$

Problem 1.20. *A compound tube consists of a steel tube 140 mm internal diameter and 160 mm external diameter and an outer brass tube 160 mm internal diameter and 180 mm external diameter. The two tubes are of the same length. The compound tube carries an axial load of 900 kN. Find the stresses and the load carried by each tube and the amount it shortens. Length of each tube is 140 mm. Take E for steel as 2×10^5 N/mm² and for brass as 1×10^5 N/mm².*

Sol. Given :

Internal dia. of steel tube = 140 mm

External dia. of steel tube = 160 mm

∴ Area of steel tube, $A_s = \frac{\pi}{4}(160^2 - 140^2) = \mathbf{4712.4\ mm^2}$

Internal dia. of brass tube = 160 mm

External dia. of brass tube = 180 mm

∴ Area of brass tube, $A_b = \frac{\pi}{4}(180^2 - 160^2) = 5340.7\ mm^2$

Axial load carried by compound tube,

$P = 900\ kN = 900 \times 1000 = 900000\ N$

Length of each tube, $L = 140$ mm

E for steel, $E_s = 2 \times 10^5\ N/mm^2$

E for brass, $E_b = 1 \times 10^5\ N/mm^2$

Let σ_s = Stress in steel in N/mm² and

σ_b = Stress in brass in N/mm²

Now strain in steel = Strain in brass

$$\frac{\sigma_s}{E_s} = \frac{\sigma_b}{E_b} \qquad \left(\because \text{ Strain} = \frac{\text{Stress}}{E}\right)$$

$$\therefore \quad \sigma_s = \frac{E_s}{E_b} \times \sigma_b = \frac{2 \times 10^5}{1 \times 10^5}\sigma_b = 2\sigma_b \qquad ...(i)$$

Now load on steel + Load on brass = Total load

or $\sigma_s \times A_s + \sigma_b \times A_b = 900000$ $\quad (\because$ Load = Stress × Area)

or $2\sigma_b \times 4712.4 + \sigma_b \times 5340.7 = 900000$ $\quad (\because \sigma_s = 2\sigma_b)$

or $14765.5\ \sigma_b = 900000$

$$\therefore \quad \sigma_b = \frac{900000}{14765.5} = \mathbf{60.95\ N/mm^2.}\ \textbf{Ans.}$$

Substituting the value of p_b in equation (*i*), we get

$$\sigma_s = 2 \times 60.95 = \mathbf{121.9\ N/mm^2.}\ \textbf{Ans.}$$

Load carried by brass tube

= Stress × Area

$= \sigma_b \times A_b = 60.95 \times 5340.7$ N

$= 325515\ N = \mathbf{325.515\ kN.}$ **Ans.**

Load carried by steel tube

$= 900 - 325.515 = \mathbf{574.485\ kN.}$ **Ans.**

Decrease in the length of the compound tube

$= \text{Decrease in length of either of the tubes}$

$= \text{Decrease in length of brass tube}$

$= \text{Strain in brass tube} \times \text{Original length}$

$$= \frac{\sigma_b}{E_b} \times L = \frac{60.95}{1 \times 10^5} \times 140 = \mathbf{0.0853\ mm.}\quad \textbf{Ans.}$$

Problem 1.22. *A load of 2 MN is applied on a short concrete column 500 mm × 500 mm. The column is reinforced with four steel bars of 10 mm diameter, one in each corner. Find the stresses in the concrete and steel bars. Take E for steel as 2.1 × 10⁵ N/mm² and for concrete as 1.4 × 10⁴ N/mm².*

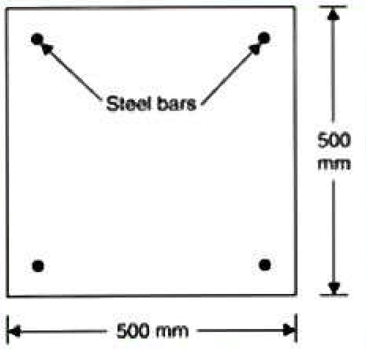


Fig. 1.18

Sol. Given :

Total load applied, $P = 2\ \text{MN} = 2 \times 10^6\ \text{N}$

Area of column $= 500 \times 500 = 250000\ \text{mm}^2$

Area of 4 steel bars, $A_s = 4 \times \frac{\pi}{4}(10)^2 = 314.159\ \text{mm}^2$

Area of concrete, A_c = Area of column − Area of steel bars

$= 250000 - 314.159$

$= 249685.841\ \text{mm}^2$

E for steel, $E_s = 2.1 \times 10^5\ \text{N/mm}^2$

E for concrete, $E_c = 1.4 \times 10^4\ \text{N/mm}^2$

Let σ_s = Stress in steel bar in N/mm²

σ_c = Stress in concrete in N/mm²

Now strain in steel = Strain in concrete

$$\frac{\sigma_s}{E_s} = \frac{\sigma_c}{E_c} \qquad \left(\because \text{Strain} = \frac{\text{Stress}}{E}\right)$$

$$\therefore \quad \sigma_s = \frac{E_s}{E_c} \times \sigma_c = \frac{2.1 \times 10^5}{1.4 \times 10^4}\sigma_c = 15\,\sigma_c \qquad ...(i)$$

Now load on steel + Load on concrete = Total load

$$\sigma_s \cdot A_s + \sigma_c \cdot A_c = P \qquad (\because \text{Load} = \text{Stress} \times \text{Area})$$

or $15\sigma_c \times 314.159 + \sigma_c \times 249685.841 = 2000000 \qquad (\because \sigma_s = 15\sigma_c)$

or $254398\sigma_c = 2000000$

$$\therefore \quad \sigma_c = \frac{2000000}{254398} = \mathbf{7.86\ N/mm^2.}\quad \textbf{Ans.}$$

Substituting this value in equation (*i*), we get

$$\sigma_s = 15 \times 7.86 = \mathbf{117.92\ N/mm^2.}\quad \textbf{Ans.}$$

THERMAL STRESSES

Thermal stresses are the stresses induced in a body due to change in temperature. Thermal stresses are set up in a body, when the temperature of the body is raised or lowered and the body is not allowed to expand or contract freely. But if the body is allowed to expand or contract freely, no stresses will be set up in the body.

Consider a body which is heated to a certain temperature.

Let L = Original length of the body,

T = Rise in temperature,

E = Young's Modulus,

α = Co-efficient of linear expansion,

dL = Extension of rod due to rise of temperature.

If the rod is free to expand, then extension of the rod is given by

$$dL = \alpha.\ T.L. \qquad ...(1.13)$$

This is shown in Fig. 1.23 (*a*) in which *AB* represents the original length and *BB'* represents the increase in length due to temperature rise. Now suppose that an external compressive load, *P* is applied at *B'* so that the rod is decreased in its length from $(L + \alpha TL)$ to *L* as shown in Figs. 1.23 (*b*) and (*c*).

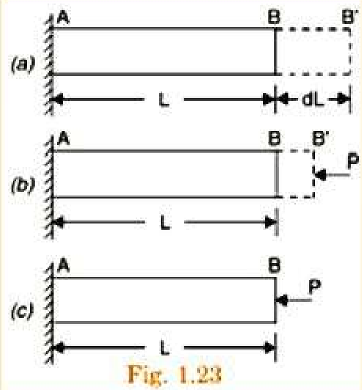


Fig. 1.23

Then compressive strain $= \dfrac{\text{Decrease in length}}{\text{Original length}}$

$$= \frac{\alpha.T.L}{L + \alpha.T.L} \approx \frac{\alpha TL}{L} = \alpha.T$$

But $\dfrac{\text{Stress}}{\text{Strain}} = E$

$\therefore$ Stress = Strain × $E = \alpha.T.E$

And load or thrust on the rod = Stress × Area = $\alpha.T.E \times A$

If the ends of the body are fixed to rigid supports, so that its expansion is prevented, then compressive stress and strain will be set up in the rod. These stresses and strains are known as thermal stresses and thermal strain.

$\therefore$ Thermal strain, $e = \dfrac{\text{Extension prevented}}{\text{Original length}}$

$$= \frac{dL}{L} = \frac{\alpha.T.L}{L} = \alpha.T \qquad ...(1.14)$$

And thermal stress, σ = Thermal strain × E

$$= \alpha.T.E. \qquad ...(1.15)$$

Thermal stress is also known as temperature stress.

And thermal strain is also known as temperature strain.

1.14.1. Stress and Strain when the Supports Yield. If the supports yield by an amount equal to δ, then the actual expansion

= Expansion due to rise in temperature − δ

$= \alpha.T.L - \delta.$

$\therefore$ Actual strain $= \dfrac{\text{Actual expansion}}{\text{Original length}} = \dfrac{(\alpha.T.L - \delta)}{L}$

And actual stress = Actual strain × E

$$= \frac{(\alpha.T.L - \delta)}{L} \times E. \qquad ...(1.16)$$

Problem 1.28. *A rod is 2 m long at a temperature of 10°C. Find the expansion of the rod, when the temperature is raised to 80°C. If this expansion is prevented, find the stress induced in the material of the rod. Take $E = 1.0 \times 10^5$ MN/m² and $\alpha = 0.000012$ per degree centigrade.*

Sol. Given :

Length of rod, L = 2 m = 200 cm

Initial temperature, T_1 = 10°C

Final temperature, T_2 = 80°C

$\therefore$ Rise in temperature, $T = T_2 - T_1 = 80° - 10° = 70°C$

Young's Modulus, $E = 1.0 \times 10^5$ MN/m²

$= 1.0 \times 10^5 \times 10^6$ N/m² $\qquad (\because M = 10^6)$

$= 1.0 \times 10^{11}$ N/m²

Co-efficient of linear expansion, $\alpha = 0.000012$

(*i*) The expansion of the rod due to temperature rise is given by equation (1.13).

$\therefore$ Expansion of the rod $= \alpha.T.L$

$= 0.000012 \times 70 \times 200$

= 0.168 cm. Ans.

(*ii*) The stress in the material of the rod if expansion is prevented is given by equation (1.15).

∴ Thermal stress, $\sigma = \alpha . T . E$

$= 0.000012 \times 70 \times 1.0 \times 10^{11}$ N/m²

$= 84 \times 10^6$ N/m² = **84 N/mm².** Ans. ($\because$ 10^6 N/m² = 1 N/mm²)

Problem 1.29. *A steel rod of 3 cm diameter and 5 m long is connected to two grips and the rod is maintained at a temperature of 95°C. Determine the stress and pull exerted when the temperature falls to 30°C, if*

(*i*) *the ends do not yield, and*

(*ii*) *the ends yield by 0.12 cm.*

Take $E = 2 \times 10^5$ *MN/m² and* $\alpha = 12 \times 10^{-6}$/°C.

Sol. Given :

Dia. of the rod, $d = 3$ cm = 30 mm

∴ Area of the rod, $A = \frac{\pi}{4} \times 30^2 = 225\,\pi$ mm²

Length of the rod, $L = 5$ m = 5000 mm

Initial temperature, $T_1 = 95$°C

Final temperature, $T_2 = 30$°C

∴ Fall in temperature, $T = T_1 - T_2 = 95 - 30 = 65$°C

Modulus of elasticity, $E = 2 \times 10^5$ MN/m²

$= 2 \times 10^5 \times 10^6$ N/m²

$= 2 \times 10^{11}$ N/m²

Co-efficient of linear expansion, $\alpha = 12 \times 10^{-6}$/°C.

(*i*) *When the ends do not yield*

The stress is given by equation (1.15).

∴ Stress = $\alpha . T . E = 12 \times 10^{-6} \times 65 \times 2 \times 10^{11}$ N/m²

$= \mathbf{156 \times 10^6}$ **N/m² or 156 N/mm² (tensile).** Ans.

Pull in the rod = Stress × Area

$= 156 \times 225\,\pi =$ **110269.9 N.** Ans.

(*ii*) *When the ends yield by 0.12 cm*

∴ $\delta = 0.12$ cm = 1.2 mm

The stress when the ends yield is given by equation (1.16).

$$\therefore \quad \text{Stress} = \frac{(\alpha . T . L - \delta)}{L} \times E$$

$$= \frac{(12 \times 10^{-6} \times 65 \times 5000 - 1.2)}{5000} \times 2 \times 10^5 \text{ N/mm}^2$$

$$= \frac{(3.9 - 1.2)}{5000} \times 2 \times 10^5 = \mathbf{108 \text{ N/mm}^2}. \text{ Ans.}$$

Pull in the rod = Stress × Area

$= 108 \times 225\,\pi =$ **76340.7 N.** Ans.

2.5.1. Volumetric Strain of a Rectangular Bar which is Subjected to an Axial Load P in the Direction of its Length.

Consider a rectangular bar of length L, width b and depth d which is subjected to an axial load P in the direction of its length as shown in Fig. 2.2.

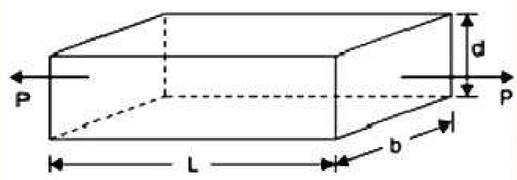


Fig. 2.2

Let δL = Change in length,

δb = Change in width, and

δd = Change in depth.

$\therefore$ Final length of the bar $= L + \delta L$

Final width of the bar $= b + \delta b$

Final depth of the bar $= d + \delta d$

Now original volume of the bar, $V = L.b.d$

Final volume $= (L + \delta L)(b + \delta b)(d + \delta d)$

$= L.b.d. + bd\delta L + Lb\delta d + Ld.\delta b$

(Ignoring products of small quantities)

$\therefore$ Change in volume,

$$\delta V = \text{Final volume} - \text{Original volume}$$
$$= (Lbd + bd\delta L + Lb\delta d + Ld\delta b) - Lbd$$
$$= bd\delta L + Lb\delta d + Ld\delta b$$

$\therefore$ Volumetric strain,

$$e_v = \frac{\delta V}{V}$$
$$= \frac{bd\delta L + Lb\delta d + Ld\delta b}{Lbd}$$
$$= \frac{\delta L}{L} + \frac{\delta d}{d} + \frac{\delta b}{b} \quad \ldots(2.4)$$

But $\frac{\delta L}{L}$ = Longitudinal strain and $\frac{\delta d}{d}$ or $\frac{\delta b}{b}$ are lateral strains.

Substituting these values in the above equation, we get

$$e_v = \text{Longitudinal strain} + 2 \times \text{Lateral strain} \quad \ldots(i)$$

From equation (2.3A), we have

$\therefore$ Lateral strain $= -\mu \times$ Longitudinal strain.

Substituting the value of lateral strain in equation (i), we get

$$e_v = \text{Longitudinal strain} - 2 \times \mu \text{ longitudinal strain}$$
$$= \text{Longitudinal strain } (1 - 2\mu)$$
$$= \frac{\delta L}{L}(1 - 2\mu) \quad \ldots(2.5)$$

Problem 2.4. *A steel bar 300 mm long, 50 mm wide and 40 mm thick is subjected to a pull of 300 kN in the direction of its length. Determine the change in volume. Take $E = 2 \times 10^5$ N/mm^2 and $\mu = 0.25$.*

Sol. Given :

Length,	L =	300 mm
Width,	b =	50 mm
Thickness,	t =	40 mm
Pull,	P =	300 kN = 300×10^3 N
Value of	E =	2×10^5 N/mm^2
Value of	μ =	0.25

Original volume, $V = L \times b \times t$

$= 300 \times 50 \times 40 \text{ mm}^3 = 600000 \text{ mm}^3$

The longitudinal strain (*i.e.*, the strain in the direction of load) is given by

$$\frac{dL}{L} = \frac{\text{Stress in the direction of load}}{E}$$

But stress in the direction of load

$$= \frac{P}{\text{Area}} = \frac{P}{b \times t}$$

$$= \frac{300 \times 10^3}{50 \times 40} = 150 \text{ N/mm}^2$$

$$\therefore \quad \frac{dL}{L} = \frac{150}{2 \times 10^5} = 0.00075$$

Now volumetric strain is given by equation (2.5) as

$$e_v = \frac{dL}{L}(1 - 2\mu)$$

$$= 0.00075\,(1 - 2 \times 0.25) = 0.000375$$

Let δV = Change in volume. Then $\frac{dV}{V}$ represents volumetric strain.

$$\therefore \quad \frac{dV}{V} = 0.000375$$

or $\quad dV = 0.000375 \times V$

$= 0.000375 \times 600000 = \mathbf{225\ mm^3}$. **Ans.**

2.7. BULK MODULUS

When a body is subjected to the mutually perpendicular like and equal direct stresses, the ratio of direct stress to the corresponding volumetric strain is found to be constant for a given material when the deformation is within a certain limit. This ratio is known as bulk modulus and is usually denoted by *K*. Mathematically bulk modulus is given by

$$K = \frac{\text{Direct stress}}{\text{Volumetric strain}} = \frac{\sigma}{\left(\frac{dV}{V}\right)} \quad \text{...(2.9)}$$

2.8. EXPRESSION FOR YOUNG'S MODULUS IN TERMS OF BULK MODULUS

Fig. 2.7 shows a cube *A B C D E F G H* which is subjected to three mutually perpendicular tensile stresses of equal intensity.

Let L = Length of cube

dL = Change in length of the cube

E = Young's modulus of the material of the cube

σ = Tensile stress acting on the faces

μ = Poisson's ratio.

Then volume of cube, $V = L^3$

Now let us consider the strain of one of the sides of the cube (say *AB*) under the action of the three mutually perpendicular stresses. This side will suffer the following three strains :

1. Strain of *AB* due to stresses on the faces *AEHD* and *BFGC*. This strain is tensile and is equal to $\frac{\sigma}{E}$.

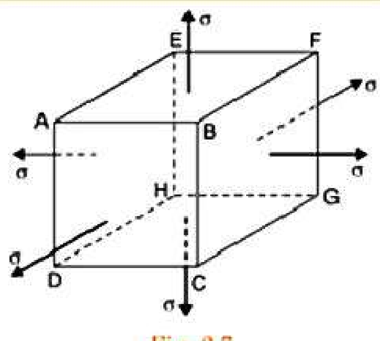


Fig. 2.7

2. Strain of AB due to stresses on the faces $AEFB$ and $DHGC$. This is compressive lateral strain and is equal to $-\mu\frac{\sigma}{E}$.

3. Strain of AB due to stresses on the faces $ABCD$ and $EFGH$. This is also compressive lateral strain and is equal to $-\mu\frac{\sigma}{E}$.

Hence the total strain of AB is given by

$$\frac{dL}{L}=\frac{\sigma}{E}-\mu\times\frac{\sigma}{E}-\mu\times\frac{\sigma}{E}=\frac{\sigma}{E}(1-2\mu) \quad ...(i)$$

Now original volume of cube, $V=L^3$ ...(*ii*)

If dL is the change in length, then dV is the change in volume.

Differentiating equation (*ii*), with respect to L,

$$dV=3L^2\times dL \quad ...(iii)$$

Dividing equation (*iii*) by equation (*ii*), we get

$$\frac{dV}{V}=\frac{3L^2\times dL}{L^3}=\frac{3dL}{L}$$

Substituting the value of $\frac{dL}{L}$ from equation (*i*), in the above equation, we get

$$\frac{dV}{V}=\frac{3\sigma}{E}(1-2\mu)$$

From equation (2.9), bulk modulus is given by

$$K=\frac{\sigma}{\left(\frac{dV}{V}\right)}=\frac{\sigma}{\frac{3\sigma}{E}(1-2\mu)} \qquad \left[\because \frac{dV}{V}=\frac{3\sigma}{E}(1-2\mu)\right]$$

$$=\frac{E}{3(1-2\mu)} \quad ...(2.10)$$

or $$E=3K(1-2\mu) \quad ...(2.11)$$

From equation (2.11), the expression for Poisson's ratio (μ) is obtained as $\mu=\frac{3K-E}{6K}$.

Problem 2.8. *For a material, Young's modulus is given as 1.2×10^5 N/mm^2 and Poisson's ratio $\frac{1}{4}$. Calculate the Bulk modulus.*

Sol. Given : Young's modulus, $E=1.2\times10^5$ N/mm^2

Poisson's ratio, $\mu=\frac{1}{4}$

Let K = Bulk modulus

Using equation (2.10),

$$K=\frac{E}{3(1-2\mu)}=\frac{1.2\times10^5}{3\left(1-\frac{2}{4}\right)}=\frac{1.2\times10^5}{3\times\frac{1}{2}}$$

$$=\frac{2\times1.2\times10^5}{3}=\mathbf{0.8\times10^5\ N/mm^2}. \quad \text{Ans.}$$

Problem 2.9. *A bar of 30 mm diameter is subjected to a pull of 60 kN. The measured extension on gauge length of 200 mm is 0.1 mm and change in diameter is 0.004 mm. Calculate :*

(i) Young's modulus, *(ii) Poisson's ratio and*

(iii) Bulk modulus.

Sol. Given : Dia. of bar, $d = 30$ mm

∴ Area of bar, $A = \frac{\pi}{4}(30)^2 = 225\pi$ mm²

Pull, $P = 60$ kN $= 60 \times 1000$ N

Gauge length, $L = 200$ mm

Extension, $\delta L = 0.1$ mm

Change in dia., $\delta d = 0.004$ mm

(i) Young's modulus (E)

Tensile stress, $\sigma = \frac{P}{A} = \frac{60000}{225\pi} = 84.87$ N/mm²

Longitudinal strain $= \frac{\delta L}{L} = \frac{0.1}{200} = 0.0005$

∴ Young's modulus, $E = \frac{\text{Tensile stress}}{\text{Longitudinal strain}}$

$= \frac{84.87}{0.0005} = 16.975 \times 10^4$ N/mm²

$=$ **1.6975×10^5 N/mm². Ans.**

(ii) Poisson's ratio (μ)

Poisson's ratio is given by equation (2.3) as

Poisson's ratio $(\mu) = \frac{\text{Lateral strain}}{\text{Longitudinal strain}}$

$$= \frac{\left(\frac{\delta d}{d}\right)}{0.0005} \qquad \left(\because \text{ Lateral strain} = \frac{\delta L}{d}\right)$$

$$= \frac{\left(\frac{0.004}{30}\right)}{0.0005} = \frac{0.000133}{0.0005} = \mathbf{0.266}. \textbf{ Ans.}$$

(iii) Bulk modulus (K)

Using equation (2.10), we get

$$K = \frac{E}{3(1-2\mu)} = \frac{1.6975 \times 10^5}{3(1-0.266 \times 2)}$$

$=$ **1.209×10^5 N/mm². Ans.**

PRINCIPLE OF COMPLEMENTARY SHEAR STRESSES

It states that a set of shear stresses across a plane is always accompanied by a set of balancing shear stresses (*i.e.*, of the same intensity) across the plane and normal to it.

Proof. Fig. 2.8 shows a rectangular block *ABCD*, subjected to a set of shear stresses of intensity τ on the faces *AB* and *CD*. Let the thickness of the block normal to the plane of the paper is unity.

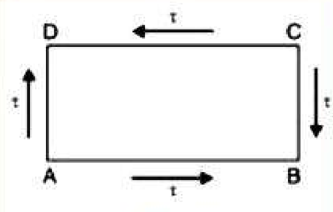


Fig. 2.8

The force acting on face *AB*

$=$ Stress × Area

$= \tau \times AB \times 1 = \tau.AB$

Similarly force acting on face *CD*

$= \tau \times CD \times 1 = \tau.CD$

$= \tau.AB \qquad (\because \; CD = AB)$

The forces acting on the faces AB and CD are equal and opposite and hence these forces will form a couple.

The moment of this couple = Force × Perpendicular distance

$$= \tau . AB \times AD \quad ...(i)$$

If the block is in equilibrium, there must be a restoring couple whose moment must be equal to the moment given by equation (i). Let the shear stress of intensity τ' is set up on the faces AD and CB.

The force acting on face $AD = \tau' \times AD \times 1 = \tau'.AD$

The force acting on face $BC = \tau' \times BC \times 1 = \tau' BC = \tau'.AD$ $\quad (\because \; BC = AD)$

As the force acting on faces AD and BC are equal and opposite, these forces also forms a couple.

Moment of this couple = Force × Distance = $\tau'.AD \times AB$ $\quad ...(ii)$

For the equilibrium of the block, the moments of couples given by equations (i) and (ii) should be equal

$$\therefore \quad \tau . AB \times AD = \tau'.AD \times AB \text{ or } \tau = \tau'.$$

The above equation proves that a set of shear stresses is always accompanied by a transverse set of shear stresses of the same intensity.

The stress τ' is known as complementary shear and the two stresses (τ and τ') at right angles together constitute a state of simple shear. The direction of the shear stresses on the block are either both towards or both away from a corner.

In Fig. 2.8, as a result of two couples, formed by the shear forces, the diagonal BD will be subjected to tension and the diagonal AB will be subjected to compression.

RELATIONSHIP BETWEEN MODULUS OF ELASTICITY AND MODULUS OF RIGIDITY

We have seen in the last article that when a square block of unit thickness is subjected to a set of shear stresses of magnitude τ on the faces AB, CD and the faces AD and CB, then the diagonal strain due to shear stress τ is given by equation (2.14) as

Total tensile strain along diagonal $BD = \frac{\tau}{E}(1+\mu)$

From equation (2.15) also we have total tensile strain in diagonal BD

$$= \frac{1}{2} \text{ shear strain} = \frac{1}{2} \times \frac{\text{Shear stress}}{C} \quad \left(\because \frac{\text{Shear stress}}{\text{Shear strain}} = \text{modulus of rigidity} = C\right)$$

$$= \frac{1}{2} \times \frac{\tau}{C} \quad (\because \text{ Shear stress} = \tau)$$

$\therefore$ Equating the two tensile strain along diagonal BD, we get

$$\frac{\tau}{E}(1+\mu) = \frac{1}{2} \times \frac{\tau}{C}$$

or
$$\frac{\tau}{E}(1+\mu) = \frac{1}{2C} \quad \text{(Cancelling } \tau \text{ from both sides)}$$

$$\therefore \quad E = 2C(1+\mu) \quad ...(2.16)$$

or
$$C = \frac{E}{2(1+\mu)} \quad ...(2.17)$$

Problem 2.10. *Determine the Poisson's ratio and bulk modulus of a material, for which Young's modulus is 1.2×10^5 N/mm² and modulus of rigidity is 4.8×10^4 N/mm².*

Sol. Given :

Young's modulus, $E = 1.2 \times 10^5 \text{ N/mm}^2$

Modulus of rigidity, $C = 4.8 \times 10^4 \text{ N/mm}^2$

Let the Poisson's ratio $= \mu$

Using equation (2.16), we get

$$E = 2C(1 + \mu)$$

or $$1.2 \times 10^5 = 2 \times 4.8 \times 10^4 (1 + \mu)$$

or $$(1 + \mu) = \frac{1.2 \times 10^5}{2 \times 4.8 \times 10^4} = 1.25 \text{ or } \mu = 1.25 - 1.0 = \mathbf{0.25}. \text{ Ans.}$$

Bulk modulus is given by equation (2.10) as

$$K = \frac{E}{3(1 - 2\mu)} = \frac{1.2 \times 10^5}{3(1 - 0.25 \times 2)} \qquad (\because \mu = 0.25)$$

$$= \mathbf{8 \times 10^4 \text{ N/mm}^2}. \text{ Ans.}$$

STRAIN ENERGY

Whenever a body is strained, the energy is absorbed in the body. The energy, which is absorbed in the body due to straining effect is known as *strain energy*. The straining effect may be due to gradually applied load or suddenly applied load or load with impact. Hence the strain energy will be stored in the body when the load is applied gradually or suddenly or with an impact. The strain energy stored in the body is equal to the work done by the applied load in stretching the body.

Before deriving the expressions for the strain energy stored in a body due to gradually applied load or suddenly applied load or load with an impact, the following terms will be defined:

1. Resilience
2. Proof resilience, and
3. Modulus of resilience.

4.2.1. Resilience. The total strain energy stored in a body is commonly known as resilience. Whenever the straining force is removed from the strained body, the body is capable of doing work. Hence the resilience is also defined as the capacity of a strained body for doing work on the removal of the straining force.

4.2.2. Proof Resilience. The maximum strain energy, stored in a body, is known as proof resilience. The strain energy stored in the body will be maximum when the body is stressed upto elastic limit. Hence the proof resilience is the quantity of strain energy stored in a body when strained upto elastic limit.

4.2.3. Modulus of Resilience. It is defined as the proof resilience of a material per unit volume. It is an important property of a material. Mathematically,

$$\text{Modulus of resilience} = \frac{\text{Proof resilience}}{\text{Volume of the body}}.$$

EXPRESSION FOR STRAIN ENERGY STORED IN A BODY WHEN THE LOAD IS APPLIED GRADUALLY

In Art. 4.1, we have mentioned that the strain energy stored in a body is equal to the work done by the applied load in stretching the body.

Fig. 4.1 shows load extension diagram of a body under tensile test upto elastic limit. The tensile load P increases gradually from zero to the value of P and the extension of the body increases from zero to the value of x.

The load P performs work in stretching the body. This work will be stored in the body as *strain energy* which is recoverable after the load P is removed.

Let P = Gradually applied load,
x = Extension of the body,
A = Cross-sectional area,
L = Length of the body,
V = Volume of the body,
E = Young's modulus,
U = Strain energy stored in the body, and
σ = Stress induced in the body.

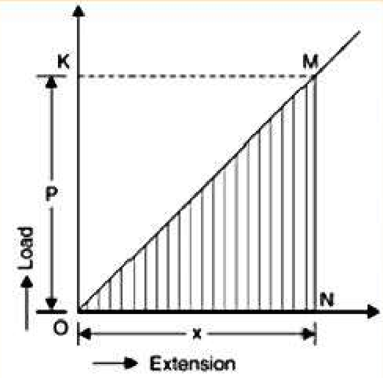


Fig. 4.1

Now work done by the load = Area of load extension curve (Shaded area in Fig. 4.1)

= Area of triangle ONM

$$= \frac{1}{2} \times P \times x. \quad ...(i)$$

But load, P = Stress × Area = $\sigma \times A$

and extension, x = Strain × Length $\left(\because \text{ Strain} = \frac{\text{Extension}}{\text{Length}} \therefore \text{Extension} = \text{Strain} \times L\right)$

$$= \frac{\text{Stress}}{E} \times L \qquad \left(\because \text{ Strain} = \frac{\text{Stress}}{E}\right)$$

$$= \frac{\sigma}{E} \times L. \quad ...(4.1)$$

Substituting the values of P and x in equation (i), we get

Work done by the load $= \frac{1}{2} \times \sigma \times A \times \frac{\sigma}{E} \times L = \frac{1}{2}\frac{\sigma^2}{E} \times A \times L$

$$= \frac{\sigma^2}{2E} \times V \qquad (\because \text{ Volume } V = A \times L)$$

But the work done by the load in stretching the body is equal to the strain energy stored in the body.

∴ Energy stored in the body,

$$U = \frac{\sigma^2}{2E} \times V. \quad ...(4.2)$$

Proof resilience. The maximum energy stored in the body without permanent deformation (*i.e.*, upto elastic limit) is known as proof resilience. Hence if in equation (4.2), the stress σ is taken at the elastic limit, we will get proof resilience.

∴ Proof resilience $= \frac{\sigma^{*2}}{2E} \times \text{Volume} \quad ...(4.3)$

where σ^* = Stress at the elastic limit.

Modulus of resilience = Strain energy per unit volume

$$= \frac{\text{Total strain energy}}{\text{Volume}} = \frac{\frac{\sigma^2}{2E} \times V}{V} = \frac{\sigma^2}{2E} \quad ...(4.4)$$

EXPRESSION FOR STRAIN ENERGY STORED IN A BODY WHEN THE LOAD IS APPLIED SUDDENLY

When the load is applied suddenly to a body, the load is constant throughout the process of the deformation of the body.

Consider a bar subjected to a sudden load.

Let P = Load applied suddenly,
L = Length of the bar,
A = Area of the cross-section,
V = Volume of the bar = $A \times L$,
E = Young's modulus,
x = Extension of the bar,
σ = Stress induced by the suddenly applied load, and
U = Strain energy stored.

As the load is applied suddenly, the load P is constant when the extension of the bar takes place.

$\therefore$ Work done by the load = Load × Extension = $P \times x$.

The maximum strain energy stored (*i.e.*, energy stored upto elastic limit) in a body is given by

$$U = \frac{\sigma^2}{2E} \times \text{Volume of the body}$$

$$= \frac{\sigma^2}{2E} \times A \times L. \qquad (\because \text{ Volume} = A \times L)$$

Equating the strain energy stored in the body to the work done, we get

$$\frac{\sigma^2}{2E} \times A \times L = P \times x = P \times \frac{\sigma}{E} \times L. \qquad \left[\because \text{ From equation (4.1), } x = \frac{\sigma}{E} \times L\right]$$

Cancelling $\frac{\sigma \times L}{E}$ from both sides, we get

$$\frac{\sigma \times A}{2} = P \quad \text{or} \quad \sigma = 2 \times \frac{P}{A}. \qquad ...(4.5)$$

From the above equation it is clear that the maximum stress induced due to suddenly applied load is twice the stress induced when the same load is applied gradually.

After obtaining the value of stress (σ), the values of extension (x) and the strain energy stored in the body may be calculated easily.

Problem 4.1. *A tensile load of 60 kN is gradually applied to a circular bar of 4 cm diameter and 5 m long. If the value of E = 2.0 × 10⁵ N/mm², determine :*

(i) stretch in the rod,

(ii) stress in the rod,

(iii) strain energy absorbed by the rod.

Sol. Given :

Gradually applied load,

$P = 60 \text{ kN} = 60 \times 1000 \text{ N}$

Dia. of rod, $d = 4 \text{ cm} = 40 \text{ mm}$

$\therefore$ Area, $A = \frac{\pi}{4} \times 40^2 = 400\,\pi \text{ mm}^2$

Length of rod, $L = 5 \text{ m} = 500 \text{ cm} = 5000 \text{ mm}$

$\therefore$ Volume of rod, $V = A \times L = 400\,\pi \times 5000 = 2 \times 10^6\,\pi \text{ mm}^3$

Young's modulus, $E = 2 \times 10^5 \text{ N/mm}^2$.

Let x = stretch or extension in the rod,
σ = stress in the rod, and
U = strain energy absorbed by the rod.

Now stress, $\sigma = \frac{\text{Load}}{\text{Area}} = \frac{P}{A} = \frac{60000}{400\pi} = \mathbf{47.746\ N/mm^2}$. **Ans.**

The stretch or extension is given by equation (4.1),

$$x = \frac{\sigma}{E} \times L = \frac{47.746}{2 \times 10^5} \times 5000 = \mathbf{1.19\ mm}. \quad \textbf{Ans.}$$

The strain energy absorbed by the rod is given by equation (4.2),

$$U = \frac{\sigma^2}{2E} \times V = \frac{47.476^2}{2 \times 2 \times 10^5} \times 2 \times 10^6 \pi = 35810 \text{ N-mm} = \mathbf{35.81\ N\text{-}m}. \quad \textbf{Ans.}$$

Problem 4.2. *If in problem 4.1, the tensile load of 60 kN is applied suddenly determine:*

(i) maximum instantaneous stress induced,

(ii) instantaneous elongation in the rod, and

(iii) strain energy absorbed in the rod.

Sol. Given :

The data given in problem 4.1 is d = 40 mm, Area = 400 π mm², L = 5000 mm, Volume = 2 × 10⁶ π mm³, E = 2 × 10⁵ N/mm² and suddenly applied load, P = 60000 N.

(i) Maximum instantaneous stress induced

Using equation (4.5),

$$\sigma = 2 \times \frac{P}{A} = 2 \times \frac{60000}{400\pi} = \mathbf{95.493\ N/mm^2}. \quad \textbf{Ans.}$$

(ii) Instantaneous elongation in the rod

Let x = Instantaneous elongation

Then $$x = \frac{\sigma}{E} \times L = \frac{95.493}{2 \times 10^5} \times 5000 \qquad \text{[see equation (4.1)]}$$

$$= \mathbf{2.38\ mm}. \quad \textbf{Ans.}$$

(iii) Strain energy is given by,

$$U = \frac{\sigma^2}{2E} \times V = \frac{95.493^2}{2 \times 2 \times 10^5} \times 2 \times 10^6 \pi = 143238 \text{ N-mm}$$

$$= \mathbf{143.238\ N\text{-}m}. \quad \textbf{Ans.}$$

Problem 4.3. *Calculate instantaneous stress produced in a bar 10 cm² in area and 3 m long by the sudden application of a tensile load of unknown magnitude, if the extension of the bar due to suddenly applied load is 1.5 mm. Also determine the suddenly applied load. Take $E = 2 \times 10^5$ N/mm².*

Sol. Given :

Area of bar, A = 10 cm² = 1000 mm²

Length of bar, L = 3 m = 3000 mm

Extension due to suddenly applied load,

x = 1.5 mm

Young's modulus, $E = 2 \times 10^5$ N/mm².

Let σ = Instantaneous stress due to sudden load, and

P = Suddenly applied load.

The extension x is given by equation (4.1),

$$x = \frac{\sigma}{E} \times L \quad \text{or} \quad 1.5 = \frac{\sigma}{2 \times 10^5} \times 3000$$

$$\therefore \quad \sigma = \frac{1.5 \times 2 \times 10^5}{3000} = \mathbf{100\ N/mm^2}. \quad \textbf{Ans.}$$

Suddenly applied load

The instantaneous stress produced by a sudden load is given by equation (4.5) as

$$\sigma = 2 \times \frac{P}{A} \quad \text{or} \quad 100 = 2 \times \frac{P}{1000}$$

$$\therefore \quad P = \frac{1000 \times 100}{2} = 50000 \text{ N} = \mathbf{50\ kN}. \quad \textbf{Ans.}$$

Problem 4.4. *A steel rod is 2 m long and 50 mm in diameter. An axial pull of 100 kN is suddenly applied to the rod. Calculate the instantaneous stress induced and also the instantaneous elongation produced in the rod. Take E = 200 GN/m².*

Sol. Given :

Length, $L = 2 \text{ m} = 2 \times 1000 = 2000 \text{ mm}$

Diameter, $d = 50 \text{ mm}$

$\therefore$ Area, $A = \frac{\pi}{4} \times 50^2 = 625\,\pi \text{ mm}^2$

Suddenly applied load,

$P = 100 \text{ kN} = 100 \times 1000 \text{ N}$

Value of $E = 200 \text{ GN/m}^2 = 200 \times 10^9 \text{ N/m}^2$ $\quad(\because \text{G} = \text{Giga} = 10^9)$

$= \frac{200 \times 10^9}{10^6} \text{ N/mm}^2$ $\quad(\because 1 \text{ m} = 1000 \text{ mm} \therefore \text{m}^2 = 10^6 \text{ mm}^2)$

$= 200 \times 10^3 \text{ N/mm}^2$

Using equation (4.5) for suddenly applied load,

$$\sigma = 2 \times \frac{P}{A} = 2 \times \frac{100 \times 1000}{625\,\pi} \text{ N/mm}^2 = \mathbf{101.86\ N/mm^2}. \quad \textbf{Ans.}$$

Let dL = Elongation

Then $$dL = \frac{P}{E} \times L = \frac{101.86}{200 \times 10^3} \times 2000 = \mathbf{1.0186\ mm}. \quad \textbf{Ans.}$$

4.5. EXPRESSION FOR STRAIN ENERGY STORED IN A BODY WHEN THE LOAD IS APPLIED WITH IMPACT

The load dropped from a certain height before the load commences to stretch the bar is a case of a load applied with impact. Consider a vertical rod fixed at the upper end and having a collar at the lower end as shown in Fig. 4.4. Let the load be dropped from a height on the collar. Due to this impact load, there will be some extension in the rod.

Let P = Load dropped (*i.e.*, load applied with impact)

L = Length of the rod,

A = Cross-sectional area of the rod,

V = Volume of rod = $A \times L$,

h = Height through which load is dropped,

δL = Extension of the rod due to load P,

E = Modulus of elasticity of the material of rod,

σ = Stress induced in the rod due to impact load.

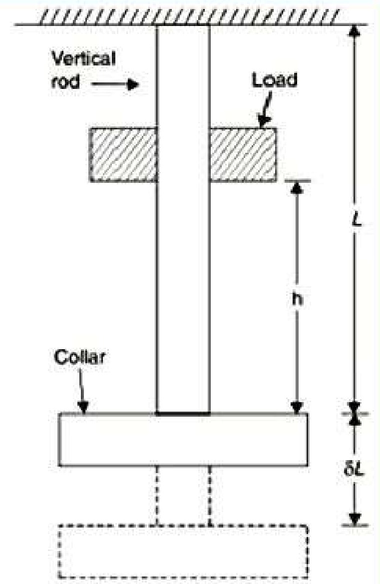


Fig. 4.4

The strain in the bar is given by,

$$\text{Strain} = \frac{\text{Stress}}{E}$$

i.e., $$\frac{\delta L}{L} = \frac{\sigma}{E}$$

$$\therefore \quad \delta L = \frac{\sigma}{E} \times L \quad \text{...(4.6)}$$

Work done by the load = Load × Distance moved

$$= P(h + \delta L) \quad ...(i)$$

The strain energy stored by the rod,

$$U = \frac{\sigma^2}{2E} \times V = \frac{\sigma^2}{2E} \times AL \quad ...(ii)$$

Equating the work done by the load to the strain energy stored, we get

$$P(h + \delta L) = \frac{\sigma^2}{2E} \cdot AL$$

or

$$P\left(h + \frac{\sigma}{E} \cdot L\right) = \frac{\sigma^2}{2E} \cdot AL \qquad \left(\because \ \delta L = \frac{\sigma}{E} \cdot L\right)$$

or

$$Ph + P \cdot \frac{\sigma}{E} \cdot L = \frac{\sigma^2}{2E} \cdot AL$$

or

$$\frac{\sigma^2}{2E} \cdot AL - P \cdot \frac{\sigma}{E} \cdot L - Ph = 0$$

Multiplying by $\frac{2E}{AL}$ to both sides, we get

$$\sigma^2 - P \cdot \frac{\sigma}{E} \cdot L \times \frac{2E}{A \cdot L} - Ph \cdot \frac{2E}{AL} = 0$$

or

$$\sigma^2 - \frac{2P}{A} \cdot \sigma - \frac{2PEh}{A \cdot L} = 0.$$

The above equation is a quadratic equation in 'σ',

$$\therefore \quad \sigma = \frac{\frac{2P}{A} \pm \sqrt{\left(\frac{2P}{A}\right)^2 + 4 \cdot \frac{2PEh}{A \cdot L}}}{2 \times 1} \qquad \left(\because \ \text{roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \frac{P}{A} \pm \sqrt{\frac{4P^2}{4A^2} + \frac{8 \cdot PEh}{4 \cdot A \cdot L}} = \frac{P}{A} \pm \sqrt{\left(\frac{P}{A}\right)^2 + \frac{2PEh}{A \cdot L}}$$

$$= \frac{P}{A} + \sqrt{\left(\frac{P}{A}\right)^2 + \frac{2PEh}{A \cdot L}} \qquad \text{(Neglecting – ve root)}$$

$$= \frac{P}{A} + \frac{P}{A}\sqrt{1 + \frac{2PEh}{A \cdot L} \times \frac{A^2}{P^2}} = \frac{P}{A} + \frac{P}{A}\sqrt{1 + \frac{2AEh}{P \cdot L}}$$

$$= \frac{P}{A}\left(1 + \sqrt{1 + \frac{2AEh}{P \cdot L}}\right) \quad ...(4.7)$$

After knowing the value of 'σ', the strain energy can be obtained.

(*i*) If δL is very small in comparison with h.

The work done by load = $P.\,h$

Equating the work done by the load to the strain energy stored in the rod, we get

$$P \cdot h = \frac{\sigma^2}{2E} \cdot AL$$

$$\therefore \quad \sigma^2 = \frac{2E \cdot P \cdot h}{A \cdot L} \quad \text{and} \quad \sigma = \sqrt{\frac{2EPh}{A \cdot L}} \quad ...(4.8)$$

(*ii*) In equation (4.7), if $h = 0$, we get

$$\sigma = \frac{P}{A}(1 + \sqrt{1 + 0}) = \frac{P}{A}(1 + 1) = \frac{2P}{A}$$

which is the case of suddenly applied load.

Once the stress σ is known, the corresponding instantaneous extension (δL) and the strain energy (U) can be obtained.

Problem 4.9. *A weight of 10 kN falls by 30 mm on a collar rigidly attached to a vertical bar 4 m long and 1000 mm² in section. Find the instantaneous expansion of the bar. Take E = 210 GPa. Derive the formula you use.*

Sol. Given :

Falling weight, $P = 10 \text{ kN} = 10{,}000 \text{ N}$

Falling height, $h = 30 \text{ mm}$

Length of bar, $L = 4 \text{ m} = 4000 \text{ mm}$

Area of bar, $A = 1000 \text{ mm}^2$

Value of $E = 210 \text{ GPa} = 210 \times 10^9 \text{ N/m}^2$ ($\because$ G = Giga = 10^9 and Pa = Pascal = 1 N/m²)

$$= \frac{210 \times 10^9 \text{ N}}{10^6 \text{ mm}^2} \quad (\because 1 \text{ m} = 1000 \text{ mm and m}^2 = 10^6 \text{ mm}^2)$$

$$= 210 \times 10^3 \text{ N/mm}^2 = 2.1 \times 10^5 \text{ N/mm}^2$$

Let dL = Instantaneous elongation due to falling weight

σ = Instantaneous stress produced due to falling weight

Using equation (4.7), we get

$$\sigma = \frac{P}{A}\left(1 + \sqrt{1 + \frac{2EAh}{P \times L}}\right)$$

$$= \frac{10000}{1000}\left(1 + \sqrt{1 + \frac{2 \times 2.1 \times 10^5 \times 1000 \times 30}{10000 \times 4000}}\right)$$

$$= 10\left(1 + \sqrt{1 + 315}\right) = 10\left(1 + \sqrt{316}\right)$$

$$= 10 \times 18.77 = 187.7 \text{ N/mm}^2$$

Now $$E = \frac{\text{Stress}}{\text{Strain}} = \frac{\sigma}{\left(\frac{\delta L}{L}\right)} \text{ or } \frac{\delta L}{L} = \frac{\sigma}{E}$$

$$\therefore \quad \delta L = \frac{\sigma}{E} \times L = \frac{187.7 \times 4000}{2.1 \times 10^5} = \mathbf{3.575 \text{ mm.}} \quad \text{Ans.}$$

Problem 4.10. *A load of 100 N falls through a height of 2 cm onto a collar rigidly attached to the lower end of a vertical bar 1.5 m long and of 1.5 cm² cross-sectional area. The upper end of the vertical bar is fixed.*

Determine :

(i) maximum instantaneous stress induced in the vertical bar,

(ii) maximum instantaneous elongation, and

(iii) strain energy stored in the vertical rod.

Take $E = 2 \times 10^5 \text{ N/mm}^2$.

Sol. Given :

Impact load, $P = 100 \text{ N}$

Height through which load falls,

$h = 2 \text{ cm} = 20 \text{ mm}$

Length of bar, $L = 1.5 \text{ m} = 1500 \text{ mm}$

Area of bar, $A = 1.5 \text{ cm}^2 = 1.5 \times 100 \text{ mm}^2 = 150 \text{ mm}^2$

$\therefore$ Volume, $V = A \times L = 150 \times 1500 = 225000 \text{ mm}^3$

Modulus of elasticity, $E = 2 \times 10^5 \text{ N/mm}^2$

Let σ = Maximum instantaneous stress induced in the vertical bar,

δL = Maximum elongation, and

U = Strain energy stored.

$$\sigma = \frac{P}{A}\left(1+\sqrt{1+\frac{2AEh}{P\,.\,L}}\right) = \frac{100}{150}\left(1+\sqrt{1+\frac{2\times 150\times 2\times 10^5\times 20}{100\times 1500}}\right)$$

$$= \frac{100}{150}(1+\sqrt{1+8000}) = \mathbf{60.23\ N/mm^2.}$$ **Ans.**

(*ii*)Using equation (4.6),

$$\delta L = \frac{\sigma}{E}\times L = \frac{60.23\times 1500}{2\times 10^5} = \mathbf{0.452\ mm.}$$ **Ans.**

(*iii*)Strain energy is given by,

$$U = \frac{\sigma^2}{2E}\times V = \frac{60.23^2}{2\times 2\times 10^5}\times 225000 = 2045 \text{ N-mm}$$

$$= \mathbf{2.045\ N\text{-}m.}$$ **Ans.**

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Strength of Material

Chapter- 02

Thin Cylinder and Spherical shell under internal Fluid Pressure

Thin Cylinder:-

The vessels such as boilers, compressed air receivers etc., are of cylindrical and spherical forms. These vessels are generally used for storing fluids (liquid or gas) under pressure. The walls of such vessels are thin as compared to their diameters. If the thickness of the wall of the cylindrical vessel is less than $\frac{1}{15}$ to $\frac{1}{20}$ of its internal diameter, the cylindrical vessel is known as a *thin cylinder*. In case of thin cylinders, the stress distribution is assumed uniform over the thickness of the wall.

Thin Cylindrical vessel subjected to internal Fluid pressure:-

Fig. 17.1 shows a thin cylindrical vessel in which a fluid under pressure is stored.

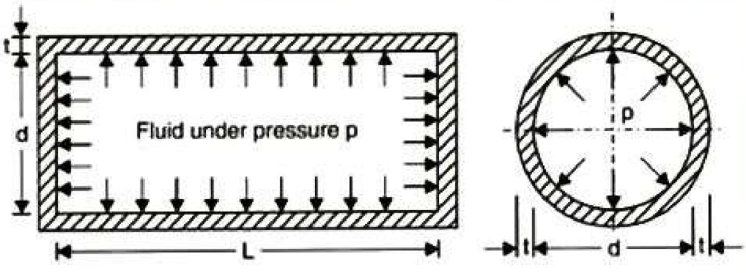


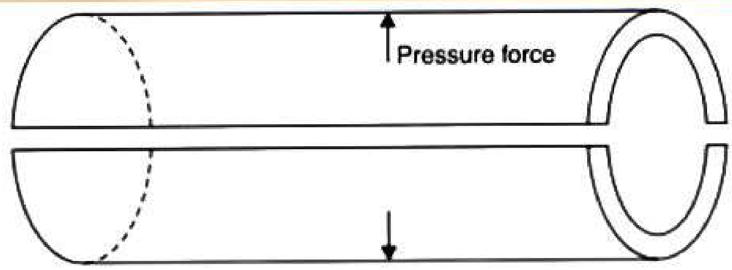
Fig. 17.1

Let d = Internal diameter of the thin cylinder
t = Thickness of the wall of the cylinder
p = Internal pressure of the fluid
L = Length of the cylinder.

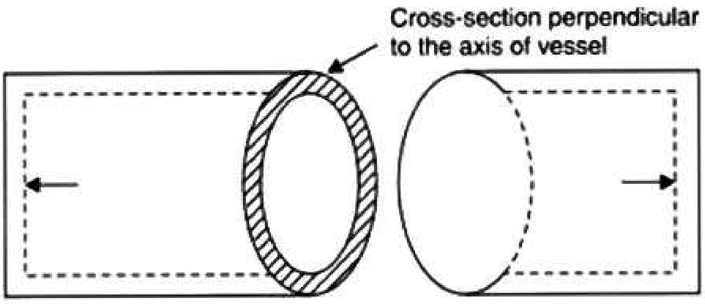
On account of the internal pressure p, the cylindrical vessel may fail by splitting up in any one of the two ways as shown in Fig. 17.2 (*a*) and 17.2 (*b*).

The forces, due to pressure of the fluid acting vertically upwards and downwards on the thin cylinder, tend to burst the cylinder as shown in Fig. 17.2 (*a*).

The forces, due to pressure of the fluid, acting at the ends of the thin cylinder, tend to burst the thin cylinder as shown in Fig. 17.2 (*b*).



(*a*)



(b)

Fig. 17.2

Stresses in thin cylindrical vessel subjected to internal Fluid pressure:-

When a thin cylindrical vessel is subjected to internal fluid pressure, the stresses in the wall of the cylinder on the cross-section along the axis and on the cross-section perpendicular to the axis are set up. These stresses are tensile and are known as :

1. Circumferential stress (or hoop stress) and
2. Longitudinal stress.

The name of the stress is given according to the direction in which the stress is acting. The stress acting along the circumference of the cylinder is called circumferential stress whereas the stress acting along the length of the cylinder (*i.e.*, in the longitudinal direction) is known as longitudinal stress. The circumferential stress is also known as *hoop stress*. The stress set up in Fig. 17.2 (*a*) is circumferential stress whereas the stress set up in Fig. 17.2 (*b*) is longitudinal stress.

Expression for Circumferential stress or Hoop Stress:-

Consider a thin cylindrical vessel subjected to an internal fluid pressure. The circumferential stress will be set up in the material of the cylinder, if the bursting of the cylinder takes place as shown in Fig. 17.3 (*a*).

The expression for hoop stress or circumferential stress (σ_1) is obtained as given below.

Let p = Internal pressure of fluid

d = Internal diameter of the cylinder

t = Thickness of the wall of the cylinder

σ_1 = Circumferential or hoop stress in the material.

The bursting will take place if the force due to fluid pressure is more than the resisting force due to circumferential stress set up in the material. In the limiting case, the two forces should be equal.

Force due to fluid pressure $= p \times$ Area on which p is acting

$= p \times (d \times L)$ ...(*i*)

($\because$ p is acting on projected area $d \times L$)

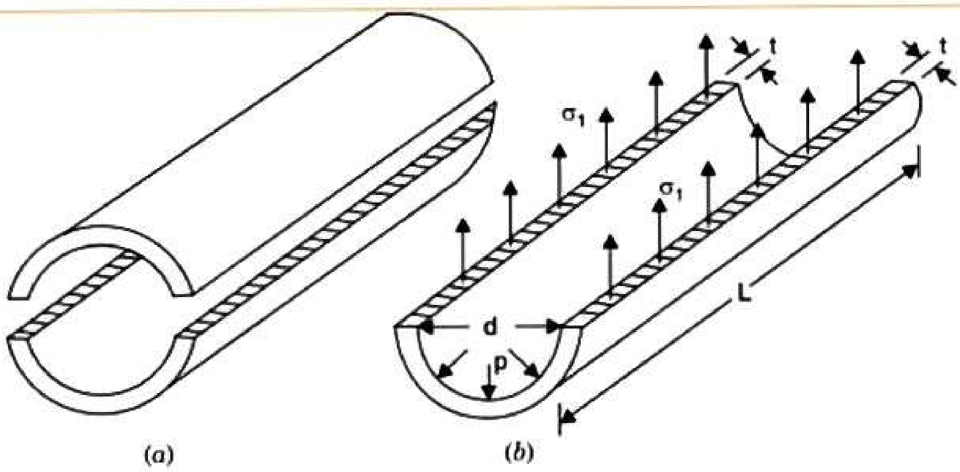


Fig. 17.3

Force due to circumferential stress

$$= \sigma_1 \times \text{Area on which } \sigma_1 \text{ is acting}$$
$$= \sigma_1 \times (L \times t + L \times t)$$
$$= \sigma_1 \times 2Lt = 2\sigma_1 \times L \times t \qquad ...(ii)$$

Equating (*i*) and (*ii*), we get

$$p \times d \times L = 2\sigma_1 \times L \times t$$

$$\therefore \qquad \sigma_1 = \frac{pd}{2t} \text{ (cancelling } L) \qquad ...(17.1)$$

This stress is tensile as shown in Fig. 17.3 (*b*).

Expression for Longitudinal Stress:-

Consider a thin cylindrical vessel subjected to internal fluid pressure. The longitudinal stress will be set up in the material of the cylinder, if the bursting of the cylinder takes place along the section *AB* of Fig. 17.4 (*a*).

The longitudinal stress (σ_2) developed in the material is obtained as :

Let p = Internal pressure of fluid stored in thin cylinder

d = Internal diameter of cylinder

t = Thickness of the cylinder

σ_2 = Longitudinal stress in the material.

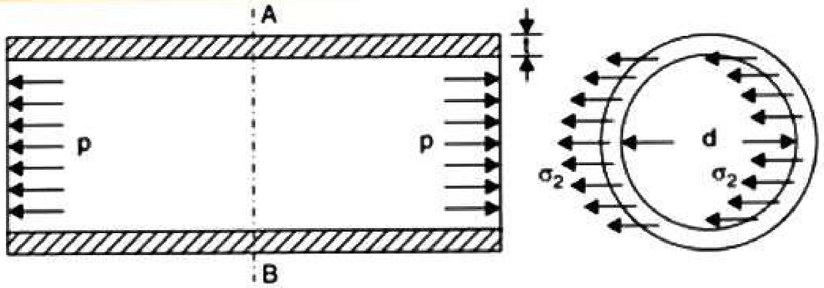
The bursting will take place if the force due to fluid pressure acting on the ends of the cylinder is more than the resisting force due to longitudinal stress (σ_2) developed in the material as shown in Fig. 17.4 (*b*). In the limiting case, both the forces should be equal.

Force due to fluid pressure $= p \times$ Area on which p is acting

$$= p \times \frac{\pi}{4} d^2$$

Resisting force $= \sigma_2 \times$ Area on which σ_2 is acting

$$= \sigma_2 \times \pi d \times t$$



(a)

(b)
Longitudinal stress (σ_2) develop

Fig. 17.4

∴ Hence in the limiting case

Force due to fluid pressure = Resisting force

$$p \times \frac{\pi}{4} d^2 = \sigma_2 \times \pi d \times t$$

∴ $$\sigma_2 = \frac{p \times \frac{\pi}{4} d^2}{\pi d \times t} = \frac{pd}{4t} \quad ...(17.2)$$

The stress σ_2 is also tensile.

Equation (17.2) can be written as

$$\sigma_2 = \frac{pd}{2 \times 2t} = \frac{1}{2} \times \sigma_1 \qquad \left(\because \ \sigma_1 = \frac{pd}{2t}\right)$$

or Longitudinal stress = Half of circumferential stress.

This also means that circumferential stress (σ_1) is two times the longitudinal stress (σ_2). Hence in the material of the cylinder the permissible stress should be less than the circumferential stress. Or in other words, the circumferential stress should not be greater than the permissible stress.

Maximum shear stress. At any point in the material of the cylindrical shell, there are two principal stresses, namely a circumferential stress of magnitude $\sigma_1 = \frac{pd}{2t}$ acting circumferentially and a longitudinal stress of magnitude $\sigma_2 = \frac{pd}{4t}$ acting parallel to the axis of the shell. These two stresses are tensile and perpendicular to each other.

∴ Maximum shear stress $$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{\frac{pd}{2t} - \frac{pd}{4t}}{2} = \frac{pd}{8t}$$

Note. (*i*) If the thickness of the thin cylinder is to be determined then equation (17.1) should be used.

(*ii*) If maximum permissible stress in the material is given. This stress should be taken circumferential stress (σ_1).

(*iii*) While using equations (17.1) and (17.2), the units of p, σ_1 and σ_2 should be same. They should be expressed either in N/mm^2 or N/m^2. Also the units of d and t should be same. They may be in metre (m) or millimetre (mm).

Problem 17.1. *A cylindrical pipe of diameter 1.5 m and thickness 1.5 cm is subjected to an internal fluid pressure of 1.2 N/mm². Determine :*

(i) Longitudinal stress developed in the pipe, and

(ii) Circumferential stress developed in the pipe.

Sol. Given :

Dia. of pipe, $d = 1.5$ m

Thickness, $t = 1.5 \text{ cm} = 1.5 \times 10^{-2}$ m

Internal fluid pressure, $p = 1.2 \text{ N/mm}^2$

As the ratio $\frac{t}{d} = \frac{1.5 \times 10^{-2}}{1.5} = \frac{1}{100}$, which is less than $\frac{1}{20}$, hence this is a case of thin cylinder.

Here unit of pressure (p) is in N/mm². Hence the unit of σ_1 and σ_2 will also be in N/mm².

(*i*) The longitudinal stress (σ_2) is given by equation (17.2) as,

$$\sigma_2 = \frac{p \times d}{4t}$$

$$= \frac{1.2 \times 1.5}{4 \times 1.5 \times 10^{-2}} = \mathbf{30\ N/mm^2}.\ \textbf{Ans.}$$

(*ii*) The circumferential stress (σ_1) is given by equation (17.1) as

$$\sigma_1 = \frac{pd}{2t}$$

$$= \frac{1.2 \times 1.5}{2 \times 1.5 \times 10^{-2}} = \mathbf{60\ N/mm^2}.\ \textbf{Ans.}$$

Problem 17.2. *A cylinder of internal diameter 2.5 m and of thickness 5 cm contains a gas. If the tensile stress in the material is not to exceed 80 N/mm², determine the internal pressure of the gas.*

Sol. Given :

Internal dia. of cylinder, $d = 2.5$ m

Thickness of cylinder, $t = 5 \text{ cm} = 5 \times 10^{-2}$ m

Maximum permissible stress $= 80 \text{ N/mm}^2$

As maximum permissible stress is given. Hence this should be equal to circumferential stress (σ_1).

We know that the circumferential stress should not be greater than the maximum permissible stress. Hence take circumferential stress equal to maximum permissible stress.

$\therefore$ $\sigma_1 = 80 \text{ N/mm}^2$

Let p = Internal pressure of the gas

Using equation (17.1),

$$\sigma_1 = \frac{pd}{2t}$$

or

$$p = \frac{2t \times \sigma_1}{d} = \frac{2 \times 5 \times 10^{-2} \times 80}{2.5}$$

(Here unit of σ_1 is in N/mm², hence unit of p will also be in N/mm²)

$$= \mathbf{3.2\ N/mm^2}.\ \textbf{Ans.}$$

Problem 17.4. *A thin cylinder of internal diameter 1.25 m contains a fluid at an internal pressure of 2 N/mm². Determine the maximum thickness of the cylinder if :*

(i) The longitudinal stress is not to exceed 30 N/mm².

(ii) The circumferential stress is not to exceed 45 N/mm².

Sol. Given :

Internal dia. of cylinder, $d = 1.25$ m

Internal pressure of fluid, $p = 2$ N/mm²

Longitudinal stress, $\sigma_2 = 30$ N/mm²

Circumferential stress, $\sigma_1 = 45$ N/mm²

Using equation (17.1),

$$\sigma_1 = \frac{pd}{2t}$$

$$\therefore \quad t = \frac{p \times d}{2 \times \sigma_1} = \frac{2 \times 1.25}{2 \times 45} = 0.0277 \text{ m}$$

$$= 2.77 \text{ cm.} \qquad \text{...(i)}$$

Using equation (17.2),

$$\sigma_2 = \frac{pd}{4t}$$

$$\therefore \quad t = \frac{pd}{4 \times \sigma_2} = \frac{2 \times 1.25}{4 \times 30} = 0.0208 \text{ m}$$

$$= 2.08 \text{ cm.} \qquad \text{...(ii)}$$

The longitudinal or circumferential stresses induced in the material are inversely proportional to the thickness (t) of the cylinder. Hence the stress induced will be less if the value of 't' is more. Hence take the maximum value of 't' calculated in equations (i) and (ii)

From equations (i) and (ii) it is clear that t should not be less than 2.77 cm.

Take t = **2.80 cm. Ans.**

Problem 17.5. *A water main 80 cm diameter contains water at a pressure head of 100 m. If the weight density of water is 9810 N/m³, find the thickness of the metal required for the water main. Given the permissible stress as 20 N/mm².*

Sol. Given :

Dia. of main, $d = 80$ cm

Pressure head of water, $h = 100$ m

Weight density of water, $w = \rho \times g = 1{,}000 \times 9.81 = 9810$ N/m³

Permissible stress $= 20$ N/mm²

Permissible stress is equal to circumferential stress (σ_1)

or $\sigma_1 = 20$ N/mm²

Pressure of water inside the water main,

$$p = \rho \times g \times h = wh = 9810 \times 100 \text{ N/m}^2$$

Here σ_1 is in N/mm², hence pressure (p) should also be N/mm². The value of p in N/mm² is given as

$$p = \frac{9810 \times 100}{1000^2 \text{ mm}^2} \text{ N/mm}^2 \qquad (\because \ 1 \text{ m} = 1000 \text{ mm})$$

$$= 0.981 \text{ N/mm}^2$$

Let t = Thickness of the metal required.

$$\sigma_1 = \frac{p \times d}{2 \times t}$$ (Here 'd' is in cm hence 't' will also be in cm)

$$t = \frac{p \times d}{2 \times \sigma_1} = \frac{0.981 \times 80}{2 \times 20} = 2 \text{ cm.}$$ **Ans.**

EFFECT OF INTERNAL PRESSURE ON THE DIMENSIONS OF A THIN CYLINDRICAL SHELL

When a fluid having internal pressure (p) is stored in a thin cylindrical shell, due to internal pressure of the fluid the stresses set up at any point of the material of the shell are :

(*i*) Hoop or circumferential stress (σ_1), acting on longitudinal section.

(*ii*) Longitudinal stress (σ_2) acting on the circumferential section.

These stresses are principal stresses, as they are acting on principal planes. The stress in the third principal plane is zero as the thickness (t) of the cylinder is very small. Actually the stress in the third principal plane is radial stress which is very small for thin cylinders and can be neglected.

Let p = Internal pressure of fluid
L = Length of cylindrical shell
d = Diameter of the cylindrical shell
t = Thickness of the cylindrical shell
E = Modulus of Elasticity for the material of the shell
σ_1 = Hoop stress in the material
σ_2 = Longitudinal stress in the material
μ = Poisson's ratio
δd = Change in diameter due to stresses set up in the material
δL = Change in length
δV = Change in volume.

The values of σ_1 and σ_2 are given by equations (17.1) and (17.2) as

$$\sigma_1 = \frac{pd}{2t}$$

$$\sigma_2 = \frac{p \times d}{4t}$$

Let e_1 = Circumferential strain,
e_2 = Longitudinal strain.

Then **circumferential strain,**

$$e_1 = \frac{\sigma_1}{E} - \frac{\mu\sigma_2}{E}$$

$$= \frac{pd}{2tE} - \frac{\mu pd}{4tE} \qquad \left(\because \sigma_1 = \frac{pd}{2t} \text{ and } \sigma_2 = \frac{pd}{4t}\right)$$

$$= \frac{pd}{2tE}\left[1 - \frac{\mu}{2}\right] \qquad \text{...(17.6)}$$

and **longitudinal strain,**

$$e_2 = \frac{\sigma_2}{E} - \frac{\mu\sigma_1}{E} \quad ...(17.7)$$

$$= \frac{pd}{4tE} - \frac{\mu pd}{2tE} \quad \text{(substituting values of } \sigma_1 \text{ and } \sigma_2\text{)}$$

$$= \frac{pd}{2tE}\left(\frac{1}{2} - \mu\right) \quad ...(17.8)$$

But circumferential strain is also given as,

$$e_1 = \frac{\text{Change in circumference due to pressure}}{\text{Original circumference}}$$

$$= \frac{\text{Final circumference} - \text{Original circumference}}{\text{Original circumference}}$$

$$= \frac{\pi(d + \delta d) - \pi d}{\pi d}$$

$$= \frac{\pi d + \pi\delta d - \pi d}{\pi d} = \frac{\pi\delta d}{\pi d}$$

$$= \frac{\delta d}{d}\left(\text{or} = \frac{\text{Change in diameter}}{\text{Original diameter}}\right) \quad ...(17.9)$$

Equating the two values of e_1 given by equations (17.6) and (17.9), we get

$$\frac{\delta d}{d} = \frac{pd}{2tE}\left[1 - \frac{\mu}{2}\right] \quad ...(17.10)$$

∴ Change in diameter,

$$\delta d = \frac{pd^2}{2tE}\left(1 - \frac{\mu}{2}\right) \quad ...(17.11)$$

Similarly longitudinal strain is also given as,

$$e_2 = \frac{\text{Change in length due to pressure}}{\text{Original length}}$$

$$= \frac{\delta L}{L} \quad ...(17.12)$$

Equating the two values of e_2 given by equations (17.8) and (17.12).

$$\therefore \quad \frac{\delta L}{L} = \frac{pd}{2tE}\left(\frac{1}{2} - \mu\right) \quad ...(17.13)$$

∴ Change in length,

$$\delta L = \frac{p \times d \times L}{2tE}\left(\frac{1}{2} - \mu\right) \quad ...(17.14)$$

Volumetric strains. It is defined as change in volume divided by original volume.

∴ Volumetric strain $= \dfrac{\delta V}{V}$

But change in volume (δV) = Final volume − Original volume

Original volume (V) = Area of cylindrical shell × Length

$$= \frac{\pi}{4} d^2 \times L$$

Final volume $= $ (Final area of cross-section) × Final length

$$= \frac{\pi}{4}[d + \delta d]^2 \times [L + \delta L]$$

$$= \frac{\pi}{4}[d^2 + (\delta d)^2 + 2d\,\delta d] \times [L + \delta L]$$

$$= \frac{\pi}{4}[d^2 L + (\delta d)^2 L + 2d\,L\delta d + \delta L d^2 + \delta L\,(\delta d)^2 + 2d\,\delta d\delta L]$$

Neglecting the smaller quantities such as $(\delta d)^2 L$, $\delta L(\delta d)^2$ and $2d\,\delta d\delta L$, we get

Final volume $= \frac{\pi}{4}[d^2 L + 2d\,L\delta d + \delta L\,d^2]$

∴ Change in volume (δV)

$$= \frac{\pi}{4}[d^2 L + 2\,dL\delta d + \delta L d^2] - \frac{\pi}{4} d^2 \times L$$

$$= \frac{\pi}{4}[2d\,L\delta d + \delta L d^2]$$

∴ Volumetric strains $= \frac{\delta V}{V} = \frac{\frac{\pi}{4}[2d\,L\delta d + \delta L d^2]}{\frac{\pi}{4} d^2 \times L}$

$$= \frac{2\delta d}{d} + \frac{\delta L}{L} \quad \text{...(17.15)}$$

$$= 2\,e_1 + e_2 \qquad \left(\because \frac{\delta d}{d} = e_1, \frac{\delta L}{L} = e_2\right) \quad \text{...(17.16)}$$

$$= 2 \times \frac{pd}{2Et}\left[1 - \frac{\mu}{2}\right] + \frac{pd}{2Et}\left(\frac{1}{2} - \mu\right)$$

(Substituting the values of e_1 and e_2)

$$= \frac{pd}{2Et}\left(2 - \frac{2\mu}{2} + \frac{1}{2} - \mu\right)$$

$$= \frac{pd}{2Et}\left(2 + \frac{1}{2} - \mu - \mu\right)$$

$$= \frac{pd}{2Et}\left(\frac{5}{2} - 2\mu\right) \quad \text{...(17.17)}$$

Also change in volume $(\delta V) = V(2e_1 + e_2)$. ...(17.18)

Problem 17.9. *Calculate : (i) the change in diameter, (ii) change in length and (iii) change in volume of a thin cylindrical shell 100 cm diameter, 1 cm thick and 5 m long when subjected to internal pressure of 3 N/mm². Take the value of $E = 2 \times 10^5$ N/mm² and Poisson's ratio, $\mu = 0.3$.*

Sol. Given :

Diameter of shell, $d = 100$ cm

Thickness of shell, $t = 1$ cm

Length of shell, $L = 5\text{ m} = 5 \times 100 = 500$ cm

Internal pressure, $p = 3$ N/mm²

Young's modulus, $E = 2 \times 10^5$ N/mm²

Poisson's ratio, $\mu = 0.30$

(i) Change in diameter (δd) is given by equation (17.11) as

$$\delta d = \frac{pd^2}{2tE}\left[1 - \frac{\mu}{2}\right]$$

$$= \frac{3 \times 100^2}{2 \times 1 \times 2 \times 10^5}\left[1 - \frac{1}{2} \times 0.30\right]$$

$$= \frac{3}{40}\,[1 - 0.15] = \mathbf{0.06375\ cm.}$$ **Ans.**

(ii) Change in length (δL) is given by equation (17.14) as

$$\delta L = \frac{pdL}{2tE}\left[\frac{1}{2} - \mu\right]$$

$$= \frac{3 \times 100 \times 500}{2 \times 1 \times 2 \times 10^5}\left[\frac{1}{2} - 0.30\right]$$

$$= \frac{15}{40} \times 0.20 = \mathbf{0.075\ cm.}$$ **Ans.**

(iii) Change in volume (δV) is given by equation (17.18) as

$$\delta V = V\,[2e_1 + e_2]$$

$$= V\left[2\frac{\delta d}{d} + \frac{\delta L}{L}\right] \qquad \left(\because\ e_1 = \frac{\delta d}{d}, e_2 = \frac{\delta L}{L}\right)$$

Substituting the values of δd, δL, d and L, we get

$$\delta V = V\left[2 \times \frac{0.06375}{100} + \frac{0.075}{500}\right]$$

$$= V\,[0.001275 + 0.00015] = 0.001425\,V.$$

But $\qquad V = \text{Original volume} = \frac{\pi}{4}\,d^2 L$

$$= \frac{\pi}{4} \times 100^2 \times 500\ \text{cm}^3 = 3926990.817\ \text{cm}^3$$

$\therefore \qquad \delta V = 0.001425 \times 3926990.817 = \mathbf{5595.96\ cm^3}.$ **Ans.**

Problem 17.10. *A cylindrical thin drum 80 cm in diameter and 3 m long has a shell thickness of 1 cm. If the drum is subjected to an internal pressure of 2.5 N/mm², determine (i) change in diameter, (ii) change in length and (iii) change in volume.*

Take $E = 2 \times 10^5$ N/mm² : Poisson's ratio = 0.25.

Sol. Given :

Diameter of drum,	$d = 80$ cm
Length of drum,	$L = 3$ m $= 3 \times 100 = 300$ cm
Thickness of drum,	$t = 1$ cm
Internal pressure,	$p = 2.5$ N/mm²
Young's modulus,	$E = 2 \times 10^5$ N/mm²
Poisson's ratio,	$\mu = 0.25$

(i) Change in diameter (δd) is given by equation (17.11) as

$$\delta d = \frac{pd^2}{2tE}\left(1 - \frac{1}{2} \times \mu\right)$$

$$= \frac{2.5 \times 80^2}{2 \times 1 \times 2 \times 10^5}\left[1 - \frac{1}{2} \times 0.25\right]$$

$$= 0.04\,[1 - 0.125] = \mathbf{0.035\ cm.}$$ **Ans.**

(*ii*) Change in length (δL) is given by equation (17.14) as

$$\delta L = \frac{pdL}{2tE}\left[\frac{1}{2} - \mu\right]$$

$$= \frac{2.5 \times 80 \times 300}{2 \times 1 \times 2 \times 10^5}\left[\frac{1}{2} - 0.25\right] = \mathbf{0.0375\ cm.} \quad \textbf{Ans.}$$

(*iii*) Using equation (17.15) for volumetric strain $\left(\frac{\delta V}{V}\right)$, we have

$$\frac{\delta V}{V} = 2\frac{\delta d}{d} + \frac{\delta L}{L}$$

$$= 2 \times \frac{0.035}{80} + \frac{0.0375}{300} \qquad \left(\because \begin{array}{ll} \delta d = 0.035, & \delta L = 0.0375 \\ d = 80, & L = 300 \end{array}\right)$$

$$= 0.000875 + 0.000125 = 0.001$$

$$\therefore \qquad \delta V = 0.001 \times V$$

where volume $V = \frac{\pi}{4} d^2 \times L = \frac{\pi}{4} \times 80^2 \times 300 = 1507964.473\ cm^3$

$\therefore$ Change in volume, $\delta V = 0.001 \times 1507964.473 = \mathbf{1507.96\ cm^3}$. **Ans.**

Problem 17.11. *A cylindrical shell 90 cm long 20 cm internal diameter having thickness of metal as 8 mm is filled with fluid at atmospheric pressure. If an additional 20 cm³ of fluid is pumped into the cylinder, find (i) the pressure exerted by the fluid on the cylinder and (ii) the hoop stress induced. Take E = 2 × 10⁵ N/mm² and μ = 0.3.*

Sol. Given :

Length of cylinder, $L = 90$ cm

Diameter of cylinder, $d = 20$ cm

Thickness of cylinder, $t = 8$ mm $= 0.8$ cm

Volume of additional fluid $= 20\ cm^3$

Volume of cylinder, $V = \frac{\pi}{4} d^2 \times L = \frac{\pi}{4} \times 20^2 \times 90$

$= 28274.33\ cm^3$

Increase in volume, δV = Volume of additional fluid

$= 20\ cm^3$

(*i*) Let p = Pressure exerted by fluid on the cylinder

$E = 2 \times 10^5\ N/mm^2$

$\mu = 0.3.$

Now using equation (17.16), volumetric strain is given as

$$\frac{\delta V}{V} = 2\,e_1 + e_2$$

or
$$\frac{20}{28274.33} = 2\,e_1 + e_2 \qquad \text{...}(i)$$

But e_1 and e_2 are circumferential and longitudinal strains and are given by equation (17.6) and (17.8) respectively as

$$e_1 = \frac{pd}{2Et}\left[1 - \frac{1}{2} \times \mu\right]$$

and
$$e_2 = \frac{pd}{2tE}\left(\frac{1}{2} - \mu\right).$$

Substituting these values in equation (*i*), we get

$$\frac{20}{28274.33} = \frac{2pd}{2Et}\left[1 - \frac{1}{2} \times \mu\right] + \frac{pd}{2tE}\left[\frac{1}{2} - \mu\right]$$

$$= \frac{2p \times 20}{2 \times 2 \times 10^5 \times 0.8}\left[1 - \frac{1}{2} \times 0.3\right] + \frac{p \times 20}{0.8 \times 2 \times 10^5}\left[\frac{1}{2} - 0.3\right]$$

or
$$0.000707 = \frac{p}{8000} \times 0.85 + \frac{p}{8000} \times 0.20 = \frac{1.05p}{8000}$$

$$\therefore \quad p = \frac{0.000707 \times 8000}{1.05} = \mathbf{5.386\ N/mm^2}. \quad \textbf{Ans.}$$

(*ii*) Hoop stress (σ_1) is given by equation (17.1) as

$$\sigma_1 = \frac{pd}{2t} = \frac{5.386 \times 20}{2 \times 0.8} = \mathbf{67.33\ N/mm^2}. \quad \textbf{Ans.}$$

Problem 17.12. *A cylindrical vessel whose ends are closed by means of rigid flange plates, is made of steel plate 3 mm thick. The length and the internal diameter of the vessel are 50 cm and 25 cm respectively. Determine the longitudinal and hoop stresses in the cylindrical shell due to an internal fluid pressure of 3 N/mm². Also calculate the increase in length, diameter and volume of the vessel. Take E = 2 × 10⁵ N/mm² and μ = 0.3.*

Sol. Given :

Thickness, $t = 3$ mm $= 0.3$ cm

Length of the cylindrical vessel, $L = 50$ cm

Internal diameter, $d = 25$ cm

Internal fluid pressure, $p = 3\ \text{N/mm}^2$

Young's modulus, $E = 2 \times 10^5\ \text{N/mm}^2$

Poisson's ratio, $\mu = 0.3$

Let σ_1 = Hoop stress and

σ_2 = Longitudinal stress.

Using equation (17.1) for hoop stress,

$$\sigma_1 = \frac{p \times d}{2t} = \frac{3 \times 25}{2 \times 0.3} = \mathbf{125\ N/mm^2}. \quad \textbf{Ans.}$$

Using equation (17.2) for longitudinal stress,

$$\sigma_2 = \frac{p \times d}{4t} = \frac{3 \times 25}{4 \times 0.3} = \mathbf{62.5\ N/mm^2}. \quad \textbf{Ans.}$$

Using equation (17.5) for circumferential strain,

$$e_1 = \frac{\sigma_1}{E} - \frac{\mu \times \sigma_2}{E}$$

$$= \frac{1}{E}[\sigma_1 - \mu \times \sigma_2]$$

$$= \frac{1}{2 \times 10^5}[125 - 62.5 \times 0.3] \quad (\because \mu = 0.3, \sigma_1 = 125 \text{ and } \sigma_2 = 62.5)$$

$$= \frac{1}{2 \times 10^5}(125 - 18.75) = \frac{106.25}{2 \times 10^5}$$

$$= 53.125 \times 10^{-5}.$$

But circumferential strain is also given by equation (17.9) as

$$e_1 = \frac{\delta d}{d}$$

Equating the two values of circumferential strain e_1, we get

$$\frac{\delta d}{d} = 53.125 \times 10^{-5}$$

$\therefore$ $\delta d = 53.125 \times 10^{-5} \times d = 53.125 \times 10^{-5} \times 25 = 0.0133$ cm

$\therefore$ Increase in diameter, δd = **0.0133 cm. Ans.**

Longitudinal strain is given by equation (17.7) as

$$e_2 = \frac{\delta L}{L} = \frac{\sigma_2}{E} - \frac{\mu \times \sigma_1}{E}$$

$$= \frac{1}{E}[\sigma_2 - \mu \times \sigma_1]$$

$$= \frac{1}{2 \times 10^5}[62.5 - 125 \times 0.3] = \frac{1}{2 \times 10^5}[62.5 - 37.5]$$

$$= \frac{2.5}{2 \times 10^5} = 12.5 \times 10^{-5}$$

$\therefore$ Increase in length, $\delta L = 12.5 \times 10^{-5} \times L$

$= 12.5 \times 10^{-5} \times 50 =$ **0.00625 cm. Ans.**

Volumetric strain is given by equation (17.16), as

$$\frac{\delta V}{V} = 2\frac{\delta d}{d} + \frac{\delta l}{l}$$

$$= 2e_1 + e_2 = 2 \times 53.125 \times 10^{-5} + 12.5 \times 10^{-5}$$

$$= 106.25 \times 10^{-5} + 12.5 \times 10^{-5} = 118.75 \times 10^{-5}$$

$\therefore$ Increase in volume,

$$\delta V = 118.75 \times 10^{-5} \times V$$

$$= 118.75 \times 10^{-5} \times \frac{\pi}{4} \times 25^2 \times 50 \qquad \left(\because \text{ volume} = \frac{\pi}{4} d^2 \times L\right)$$

$=$ **29.13 cm². Ans.**

Problem 17.13. *A cylindrical vessel is 1.5 m diameter and 4 m long is closed at ends by rigid plates. It is subjected to an internal pressure of 3 N/mm². If the maximum principal stress is not to exceed 150 N/mm², find the thickness of the shell. Assume E = 2 × 10⁵ N/mm² and Poisson's ratio = 0.25. Find the changes in diameter, length and volume of the shell.*

Sol. Given :

Dia., $d = 1.5 \text{ m} = 1500 \text{ mm}$

Length, $L = 4 \text{ m} = 4000 \text{ mm}$

Internal pressure, $p = 3 \text{ N/mm}^2$

Max. principal stress $= 150 \text{ N/mm}^2$

Max. principal stress means the circumferential stress

$\therefore$ Circumferential stress, $\sigma_1 = 150 \text{ N/mm}^2$

Value of $E = 2 \times 10^5 \text{ N/mm}^2$.

Poisson's ratio, $\mu = 0.25$

Let t = thickness of the shell,

δd = change in diameter,

δL = change in length, and

δV = change in volume.

(*i*) Using equation (17.1),

$$\sigma_1 = \frac{p \times d}{2t}$$

$$\therefore \quad t = \frac{p \times d}{2 \times \sigma_1} = \frac{3 \times 1500}{2 \times 150}$$

(Here p and σ_1 are in same units, 'd' is in mm hence 't' will be in mm)

$$= \mathbf{15 \text{ mm.}} \quad \textbf{Ans.}$$

(*ii*) Using equation (17.11),

$$\delta d = \frac{pd^2}{2t \times E}\left(1 - \frac{1}{2} \times \mu\right)$$

$$= \frac{3 \times 1500^2}{2 \times 15 \times 2 \times 10^5}\left(1 - \frac{1}{2} \times 0.25\right) = \mathbf{0.984 \text{ mm.}} \quad \textbf{Ans.}$$

(*iii*) Using equation (17.14),

$$\delta L = \frac{p \times d \times L}{2t \times E}\left(\frac{1}{2} - \mu\right)$$

$$= \frac{3 \times 1500 \times 4000}{2 \times 15 \times 2 \times 10^5}\left(\frac{1}{2} - 0.25\right)$$

$$= \mathbf{0.75 \text{ mm.}} \quad \textbf{Ans.}$$

(*iv*) Using equation (17.17),

$$\frac{\delta V}{V} = \frac{p \times d}{2E \times t}\left(\frac{5}{2} - 2 \times \mu\right)$$

$$= \frac{3 \times 1500}{2 \times 2 \times 10^5 \times 15}\left(\frac{5}{2} - 2 \times 0.25\right) = \frac{3 \times 1500 \times 2}{4 \times 10^5 \times 15}$$

$$\therefore \quad \delta V = \frac{3}{2000} \times V = \frac{3}{2000} \times \left(\frac{\pi}{4} \times d^2 \times L\right)$$

$$= \frac{3}{2000} \times \left(\frac{\pi}{4} \times 1500^2 \times 4000\right) = \mathbf{10602875 \text{ mm}^3.} \quad \textbf{Ans.}$$

Problem 17.15. *A cylindrical shell 3 metres long which is closed as the ends has an internal diameter of 1 m and a wall thickness of 15 mm. Calculate the circumferential and longitudinal stresses induced and also changes in the dimensions of the shell, if it is subjected to an internal pressure of 1.5 N/mm². Take E = 2 × 10⁵ N/mm² and μ = 0.3.*

Sol. Given :

Length of shell, $L = 3 \text{ m} = 300 \text{ cm}$

Internal diameter, $d = 1 \text{ m} = 100 \text{ cm}$

Wall thickness, $t = 15 \text{ mm} = 1.5 \text{ cm}$

Internal pressure, $p = 1.5 \text{ N/mm}^2$

Young's modulus, $E = 2 \times 10^5 \text{ N/mm}^2$

Poisson's ratio, $\mu = 0.3$

Let σ_1 = Circumferential (or Hoop) stress, and

σ_2 = Longitudinal stress.

Using equation (17.1) for hoop stress,

$$\sigma_1 = \frac{pd}{2t}$$

$$= \frac{1.5 \times 100}{2 \times 1.5} = \mathbf{50 \text{ N/mm}^2}. \quad \textbf{Ans.}$$

Using equation (17.2) for longitudinal stress,

$$\sigma_2 = \frac{p \times d}{4t}$$

$$= \frac{1.5 \times 100}{4 \times 1.5} = \mathbf{25 \text{ N/mm}^2}. \quad \textbf{Ans.}$$

Changes in the dimensions

Using equation (17.11) for the change in diameter (δd),

$$\delta d = \frac{pd^2}{2tE}\left(1 - \frac{1}{2} \times \mu\right)$$

$$= \frac{1.5 \times 100^2}{2 \times 1.5 \times 2 \times 10^5}\left(1 - \frac{1}{2} \times 0.3\right) \qquad (\because \mu = 0.3)$$

$$= \frac{1}{4 \times 10^3}(1 - 0.15) = \frac{0.85}{4 \times 10^3}$$

$$= \mathbf{0.2125 \times 10^{-3} \text{ cm}}. \quad \textbf{Ans.}$$

Using equation (17.14) for change in length, we get

$$\delta L = \frac{p \times d \times L}{2tE}\left(\frac{1}{2} - \mu\right)$$

$$= \frac{1.5 \times 100 \times 300}{2 \times 1.5 \times 2 \times 10^5}\left(\frac{1}{2} - 0.3\right)$$

$$= \frac{10 \times 100 \times 300}{4 \times 10^6} \times 0.2 = \frac{0.06}{4} = \mathbf{0.015 \text{ cm}}. \quad \textbf{Ans.}$$

Using equation (17.17) for volumetric strain, we get

$$\frac{\delta V}{V} = \frac{p \times d}{2Et}\left[\frac{5}{2} - 2\mu\right]$$

$$= \frac{1.5 \times 100}{2 \times 2 \times 10^5 \times 1.5}[2.5 - 2 \times 0.3] \qquad (\because \mu = 0.3)$$

$$= 0.25 \times 10^{-3} \times [2.5 - 0.6]$$

$$= 0.25 \times 10^{-3} \times 1.9 = 0.475 \times 10^{-3}$$

$\therefore$ Change in volume, $\delta V = 0.475 \times 10^{-3} \times V$

where V = Original volume

$$= \frac{\pi}{4} d^2 \times L = \frac{\pi}{4} \times 100^2 \times 300 = 2356194.49 \text{ cm}^3.$$

$\therefore$ $$\delta V = 0.475 \times 10^{-3} \times 2356194.49 = \mathbf{1119.19 \text{ cm}^3}. \quad \textbf{Ans.}$$

Problem 17.16. *A thin cylindrical shell with following dimensions is filled with a liquid at atmospheric pressure : Length = 1.2 m, external diameter = 20 cm, thickness of metal = 8 mm.*

Find the value of the pressure exerted by the liquid on the walls of the cylinder and the hoop stress induced if an additional volume of 25 cm³ of liquid is pumped into the cylinder. Take $E = 2.1 \times 10^5$ N/mm² and Poisson's ratio = 0.33.

Sol. Given :

Length, $L = 1.2 \text{ m} = 1200 \text{ mm}$

External dia. $D = 20 \text{ cm} = 200 \text{ mm}$

Thickness, $t = 8 \text{ mm}$

$\therefore$ Internal dia., $d = D - 2 \times t = 200 - 2 \times 8 = 184 \text{ mm}$

Additional volume, $\delta V = 25 \text{ cm}^3 = 25 \times 10^3 \text{ mm}^3 = 25000 \text{ mm}^3$

Value of $E = 2.1 \times 10^5 \text{ N/mm}^2$

Poisson's ratio, $\mu = 0.33$

Let p = Pressure exerted, and

σ_1 = hoop stress produced.

Volume of liquid or inside volume of cylinder,

$$V = \frac{\pi}{4} d^2 \times L$$

$$= \frac{\pi}{4} \times 184^2 \times 1200 = 31908528 \text{ mm}^3$$

(*i*) Using equation (17.17),

$$\frac{\delta V}{V} = \frac{p \times d}{2E \times t}\left(\frac{5}{2} - 2\mu\right)$$

or

$$\frac{25000}{31908528} = \frac{p \times 184}{2 \times 2.1 \times 10^5 \times 8}\left(\frac{5}{2} - 2 \times 0.33\right)$$

$\therefore$ $$p = \frac{25000 \times 2 \times 2.1 \times 10^5 \times 8}{31908528 \times 184 \times (2.5 - 0.66)} = \mathbf{7.77 \text{ N/mm}^2}. \quad \textbf{Ans.}$$

(*ii*) Using equation (17.1),

$$\sigma_1 = \frac{p \times d}{2t} = \frac{7.77 \times 184}{2 \times 8} = \mathbf{89.42 \text{ N/mm}^2}. \quad \textbf{Ans.}$$

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Strength of Material

Chapter- 03

TWO DIMENSIONAL STRESS SYSTEM

PRINCIPAL PLANES AND PRINCIPAL STRESSES

The planes, which have no shear stress, are known as principal planes. Hence principal planes are the planes of zero shear stress. These planes carry only normal stresses.

The normal stresses, acting on a principal plane, are known as principal stresses.

METHODS FOR DETERMINING STRESSES ON OBLIQUE SECTION

The stresses on oblique section are determined by the following methods :

1. Analytical method, and 2. Graphical method.

ANALYTICAL METHOD FOR DETERMINING STRESSES ON OBLIQUE SECTION

The following two cases will be considered :

1. A member subjected to a direct stress in one plane.
2. The member is subjected to like direct stresses in two mutually perpendicular directions.

3.4.1. A Member Subjected to a Direct Stress in one Plane. Fig. 3.1 (*a*) shows a rectangular member of uniform cross-sectional area *A* and of unit thickness.

Let P = Axial force acting on the member.

A = Area of cross-section, which is perpendicular to the line of action of the force P.

The stress along x-axis, $\sigma = \dfrac{P}{A}$

Hence, the member is subjected to a stress along x-axis.

Consider a cross-section EF which is perpendicular to the line of action of the force P.

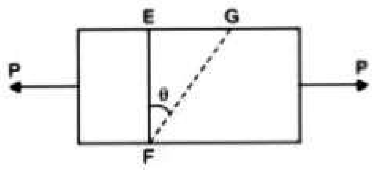

Fig. 3.1 (*a*)

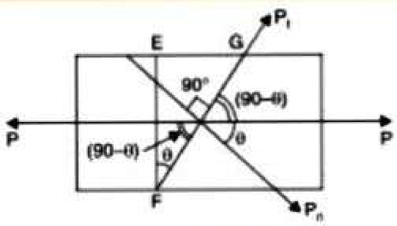

Fig. 3.1 (*b*)

Then area of section, $EF = EF \times 1 = A$.

The stress on the section EF is given by

$$\sigma = \frac{\text{Force}}{\text{Area of } EF} = \frac{P}{A} \qquad ...(i)$$

The stress on the section EF is entirely normal stress. There is no shear stress (or tangential stress) on the section EF.

Now consider a section FG at an angle θ with the normal cross-section EF as shown in Fig. 3.1 (*a*).

Area of section FG = $FG \times 1$ (member is having unit thickness)

$$= \frac{EF}{\cos\theta} \times 1 \qquad \left(\because \text{In } \Delta\, EFG, \frac{EF}{FG} = \cos\theta \therefore FG = \frac{EF}{\cos\theta}\right)$$

$$= \frac{A}{\cos\theta} \qquad (\because \; EF \times 1 = A)$$

∴ Stress on the section, FG

$$= \frac{\text{Force}}{\text{Area of section } FG} = \frac{P}{\left(\frac{A}{\cos\theta}\right)} = \frac{P}{A}\cos\theta$$

$$= \sigma\cos\theta \qquad \left(\because \frac{P}{A} = \sigma\right) \qquad \ldots(3.1)$$

This stress, on the section FG, is parallel to the axis of the member (*i.e.*, this stress is along x-axis). This stress may be resolved in two components. One component will be normal to the section FG whereas the second component will be along the section FG (*i.e.*, tangential to the section FG). The normal stress and tangential stress (*i.e.*, shear stress) on the section FG are obtained as given below [Refer to Fig. 3.1 (*b*)].

Let P_n = The component of the force P, normal to section FG
= $P\cos\theta$
P_t = The component of force P, along the surface of the section FG (or tangential to the surface FG)
= $P\sin\theta$
σ_n = Normal stress across the section FG
σ_t = Tangential stress (*i.e.*, shear stress) across the section FG.

∴ Normal stress and tangential stress across the section FG are obtained as,

Normal stress, $$\sigma_n = \frac{\text{Force normal to section } FG}{\text{Area of section } FG}$$

$$= \frac{P_n}{\left(\frac{A}{\cos\theta}\right)} = \frac{P\cos\theta}{\left(\frac{A}{\cos\theta}\right)} \qquad (\because \; P_n = P\cos\theta)$$

$$= \frac{P}{A}\cos\theta \,.\, \cos\theta = \frac{P}{A}\cos^2\theta$$

$$= \sigma\cos^2\theta \qquad \left(\because \frac{P}{A} = \sigma\right) \qquad \ldots(3.2)$$

Tangential stress (*i.e.*, shear stress),

$$\sigma_t = \frac{\text{Tangential force across section } FG}{\text{Area of section } FG}$$

$$= \frac{P_t}{\left(\frac{A}{\cos\theta}\right)} = \frac{P\sin\theta}{\left(\frac{A}{\cos\theta}\right)} \qquad (\because \; P_t = P\sin\theta)$$

$$= \frac{P}{A}\sin\theta \,.\, \cos\theta = \sigma\sin\theta \,.\, \cos\theta$$

$$= \frac{\sigma}{2} \times 2\sin\theta\cos\theta \qquad \text{[Multiplying and dividing by 2]}$$

$$= \frac{\sigma}{2}\sin 2\theta \qquad (\because \; 2\sin\theta\cos\theta = \sin 2\theta) \qquad \ldots(3.3)$$

From equation (3.2), it is seen that the normal stress (σ_n) on the section FG will be maximum, when $\cos^2 \theta$ or $\cos \theta$ is maximum. And $\cos \theta$ will be maximum when $\theta = 0°$ as $\cos 0° = 1$. But when $\theta = 0°$, the section FG will coincide with section EF. But the section EF is normal to the line of action of the loading. This means the plane normal to the axis of loading will carry the maximum normal stress.

$\therefore$ Maximum normal stress, $= \sigma \cos^2 \theta = \sigma \cos^2 0° = \sigma$ ...(3.4)

From equation (3.3), it is observed that the tangential stress (*i.e.*, shear stress) across the section FG will be maximum when $\sin 2\theta$ is maximum. And $\sin 2\theta$ will be maximum when $\sin 2\theta = 1$ or $2\theta = 90°$ or $270°$

or $\theta = 45°$ or $135°$.

This means the shear stress will be maximum on two planes inclined at 45° and 135° to the normal section EF as shown in Figs. 3.1 (*c*) and 3.1 (*d*).

$$\therefore \text{ Max. value of shear stress} = \frac{\sigma}{2} \sin 2\theta = \frac{\sigma}{2} \sin 90° = \frac{\sigma}{2}. \quad ...(3.5)$$

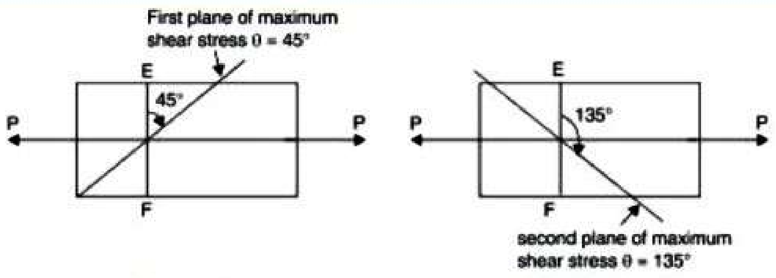

Fig. 3.1 (*c*) Fig. 3.1 (*d*)

From equations (3.4) and (3.5) it is seen that maximum normal stress is equal to σ whereas the maximum shear stress is equal to $\sigma/2$ or equal to half the value of greatest normal stress.

Problem 3.1. *A rectangular bar of cross-sectional area 10000 mm² is subjected to an axial load of 20 kN. Determine the normal and shear stresses on a section which is inclined at an angle of 30° with normal cross-section of the bar.*

Sol. Given :

Cross-sectional area of the rectangular bar,

$A = 10000 \text{ mm}^2$

Axial load, $P = 20 \text{ kN} = 20{,}000 \text{ N}$

Angle of oblique plane with the normal cross-section of the bar,

$\theta = 30°$

Now direct stress $\sigma = \dfrac{P}{A} = \dfrac{20000}{10000} = 2 \text{ N/mm}^2$

Let σ_n = Normal stress on the oblique plane

σ_t = Shear stress on the oblique plane.

Using equation (3.2) for normal stress, we get

$$\sigma_n = \sigma \cos^2 \theta$$
$$= 2 \times \cos^2 30° \quad (\because \sigma = 2 \text{ N/mm}^2)$$
$$= 2 \times 0.866^2 \quad (\because \cos 30° = 0.866)$$
$$= \mathbf{1.5 \text{ N/mm}^2}. \text{ **Ans.**}$$

Using equation (3.3) for shear stress, we get

$$\sigma_t = \frac{\sigma}{2} \sin 2\theta = \frac{2}{2} \times \sin (2 \times 30°)$$
$$= 1 \times \sin 60° = \mathbf{0.866 \text{ N/mm}^2}. \text{ **Ans.**}$$

Problem 3.2. *Find the diameter of a circular bar which is subjected to an axial pull of 160 kN, if the maximum allowable shear stress on any section is 65 N/mm².*

Sol. Given :

Axial pull, $P = 160 \text{ kN} = 160000 \text{ N}$

Maximum shear stress $= 65 \text{ N/mm}^2$

Let D = Diameter of the bar

∴ Area of the bar $= \frac{\pi}{4} D^2$

∴ Direct stress, $\sigma = \frac{P}{A} = \frac{160000}{\frac{\pi}{4} D^2} = \frac{640000}{\pi D^2} \text{ N/mm}^2$

Maximum shear stress is given by equation (3.5).

∴ Maximum shear stress $= \frac{\sigma}{2} = \frac{640000}{2 \times \pi D^2}$.

But maximum shear stress is given as = 65 N/mm².

Hence equating the two values of maximum shear, we get

$$\therefore \quad 65 = \frac{640000}{2 \times \pi D^2}$$

$$\therefore \quad D^2 = \frac{640000}{2 \times \pi \times 65} = 1567$$

$$\therefore \quad D = \mathbf{39.58 \text{ mm}}. \quad \text{Ans.}$$

Problem 3.3. *A rectangular bar of cross-sectional area of 11000 mm² is subjected to a tensile load P as shown in Fig. 3.3. The permissible normal and shear stresses on the oblique plane BC are given as 7 N/mm² and 3.5 N/mm² respectively. Determine the safe value of P.*

Sol. Given :

Area of cross-section, $A = 11000 \text{ mm}^2$

Normal stress, $\sigma_n = 7 \text{ N/mm}^2$

Shear stress, $\sigma_t = 3.5 \text{ N/mm}^2$

Angle of oblique plane with the axis of bar = 60°.

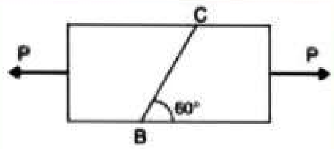

Fig. 3.3

∴ Angle of oblique plane BC with the normal cross-section of the bar,

$$\theta = 90° - 60° = 30°$$

Let P = Safe value of axial pull

σ = Safe stress in the member.

Using equation (3.2),

$$\sigma_n = \sigma \cos^2 \theta \quad \text{or} \quad 7 = \sigma \cos^2 30°$$

$$= \sigma (0.866)^2. \qquad (\because \cos 30° = 0.866)$$

$$\therefore \quad \sigma = \frac{7}{0.866 \times 0.866} = 9.334 \text{ N/mm}^2$$

Using equation (3.3),

$$\sigma_t = \frac{\sigma}{2} \sin 2\theta$$

or

$$3.5 = \frac{\sigma}{2} \sin 2 \times 30° = \frac{\sigma}{2} \sin 60° = \frac{\sigma}{2} \times 0.866$$

$$\therefore \quad \sigma = \frac{3.5 \times 2}{0.866} = 8.083 \text{ N/mm}^2.$$

The safe stress is the least of the two, *i.e.*, 8.083 N/mm².

∴ Safe value of axial pull,

$$P = \text{Safe stress} \times \text{Area of cross-section}$$

$$= 8.083 \times 11000 = 88913 \text{ N} = \mathbf{88.913 \text{ kN}}. \quad \text{Ans.}$$

3.4.2. A Member Subjected to like Direct Stresses in two Mutually Perpendicular Directions. Fig. 3.4 (*a*) shows a rectangular bar *ABCD* of uniform cross-sectional area *A* and of unit thickness. The bar is subjected to two direct tensile stresses (or two-principal tensile stresses) as shown in Fig. 3.4 (*a*).

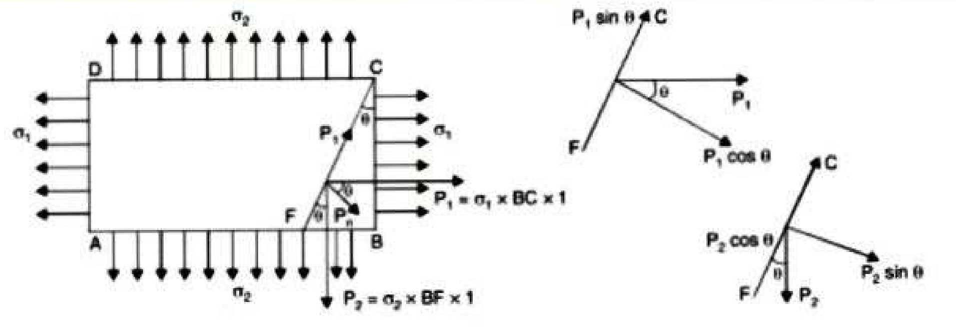

Fig. 3.4 (*a*)

Let *FC* be the oblique section on which stresses are to be calculated. This can be done by converting the stresses σ_1 (acting on face *BC*) and σ_2 (acting on face *AB*) into equivalent forces. Then these forces will be resolved along the inclined plane *FC* and perpendicular to *FC*. Consider the forces acting on wedge *FBC*.

Let θ = Angle made by oblique section *FC* with normal cross-section *BC*

σ_1 = Major tensile stress on face *AD* and *BC*

σ_2 = Minor tensile stress on face *AB* and *CD*

P_1 = Tensile force on face *BC*

P_2 = Tensile force on face *FB*.

The tensile force on face *BC*,

$$P_1 = \sigma_1 \times \text{Area of face } BC = \sigma_1 \times BC \times 1 \qquad (\because \ \text{Area} = BC \times 1)$$

The tensile force on face *FB*,

$$P_2 = \text{Stress on } FB \times \text{Area of } FB = \sigma_2 \times FB \times 1.$$

The tensile forces P_1 and P_2 are also acting on the oblique section *FC*. The force P_1 is acting in the axial direction, whereas the force P_2 is acting downwards as shown in Fig. 3.4 (*a*). Two forces P_1 and P_2 each can be resolved into two components *i.e.*, one normal to the plane *FC* and other along the plane *FC*. The components of P_1 are $P_1 \cos\theta$ normal to the plane *FC* and $P_1 \sin\theta$ along the plane in the upward direction. The components of P_2 are $P_2 \sin\theta$ normal to the plane *FC* and $P_2 \cos\theta$ along the plane in the downward direction.

Let P_n = Total force normal to section *FC*

= Component of force P_1 normal to section *FC* + Component of force P_2 normal to section *FC*

$= P_1 \cos\theta + P_2 \sin\theta$

$= \sigma_1 \times BC \times \cos\theta + \sigma_2 \times BF \times \sin\theta \quad (\because \ P_1 = \sigma_1 \times BC, P_2 = \sigma_2 \times BF)$

P_t = Total force along the section *FC*

= Component of force P_1 along the section *FC* + Component of force P_2 along the section *FC*

$= P_1 \sin\theta + (-P_2 \cos\theta)$ (–ve sign is taken due to opposite direction)

$= P_1 \sin\theta - P_2 \cos\theta$

$= \sigma_1 \times BC \times \sin\theta - \sigma_2 \times BF \times \cos\theta$

(Substituting the values P_1 and P_2)

σ_n = Normal stress across the section *FC*

$$= \frac{\text{Total force normal to the section } FC}{\text{Area of section } FC}$$

$$= \frac{P_n}{FC \times 1} = \frac{\sigma_1 \times BC \times \cos\theta + \sigma_2 \times BF \times \sin\theta}{FC}$$

$$= \sigma_1 \times \frac{BC}{FC} \times \cos\theta + \sigma_2 \times \frac{BF}{FC} \times \sin\theta$$

$$= \sigma_1 \times \cos\theta \times \cos\theta + \sigma_2 \times \sin\theta \times \sin\theta$$

$$\left(\because \text{ In triangle } FBC,\ \frac{BC}{FC} = \cos\theta,\ \frac{BF}{FC} = \sin\theta\right)$$

$$= \sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta$$

$$= \sigma_1 \left(\frac{1+\cos 2\theta}{2}\right)^{*} + \sigma_2 \left(\frac{1-\cos 2\theta}{2}\right)^{**}$$

$$[\because \cos^2\theta = (1+\cos 2\theta)/2 \text{ and } \sin^2\theta = (1-\cos 2\theta)/2]$$

$$= \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}\cos 2\theta \quad \text{...(3.6)}$$

σ_t = Tangential stress (or shear stress) along section FC

$$= \frac{\text{Total force along the section } FC}{\text{Area of section } FC} \quad \left(\because \text{ Stress} = \frac{\text{Force}}{\text{Area}}\right)$$

$$= \frac{P_t}{FC \times 1} = \frac{\sigma_1 \times BC \times \sin\theta - \sigma_2 \times BF \times \cos\theta}{FC}$$

$$= \sigma_1 \times \frac{BC}{FC} \times \sin\theta - \sigma_2 \times \frac{BF}{FC} \times \cos\theta$$

$$= \sigma_1 \times \cos\theta \times \sin\theta - \sigma_2 \times \sin\theta \times \cos\theta$$

$$\left(\because \text{ In triangle } FBC,\ \frac{BC}{FC} = \cos\theta,\ \frac{BF}{FC} = \sin\theta\right)$$

$$= (\sigma_1 - \sigma_2)\cos\theta \sin\theta$$

$$= \frac{(\sigma_1-\sigma_2)}{2} \times 2\cos\theta\sin\theta \quad \text{(Multiplying and dividing by 2)}$$

$$= \frac{(\sigma_1-\sigma_2)}{2}\sin 2\theta \quad \text{...(3.7)}$$

The resultant stress on the section FC will be given as

$$\sigma_R = \sqrt{\sigma_n^2 + \sigma_t^2} \quad \text{...(3.8)}$$

Obliquity [Refer to Fig. 3.4 (*b*)]. The angle made by the resultant stress with the normal of the oblique plane, is known as obliquity. It is denoted by ϕ. Mathematically,

$$\tan\phi = \frac{\sigma_t}{\sigma_n} \quad \text{...[3.8 (A)]}$$

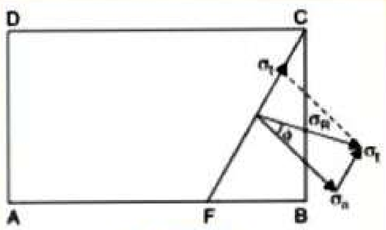

Fig. 3.4 (*b*)

Maximum shear stress. The shear stress is given by equation (3.7). The shear stress (σ_t) will be maximum when

$$\sin 2\theta = 1 \text{ or } 2\theta = 90^\circ \text{ or } 270^\circ \quad (\because \sin 90^\circ = 1 \text{ and also } \sin 270^\circ = 1)$$

or $$\theta = 45^\circ \text{ or } 135^\circ$$

And maximum shear stress, $(\sigma_t)_{max} = \frac{\sigma_1 - \sigma_2}{2}$ ...(3.9)

NOTE

$^{*}\cos 2\theta = \cos^2\theta - \sin^2\theta$ $= \cos^2\theta - (1-\cos^2\theta) = 2\cos^2\theta - 1$ $\therefore \cos^2\theta = \frac{(1+\cos 2\theta)}{2}$	$^{**}\cos 2\theta = \cos^2\theta - \sin^2\theta$ $= (1-\sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta$ $\therefore \sin^2\theta = \frac{(1-\cos 2\theta)}{2}$

The planes of maximum shear stress are obtained by making an angle of 45° and 135° with the plane *BC* (at any point on the plane *BC*) in such a way that the planes of maximum shear stress lie within the material as shown in Fig. 3.4 (*c*).

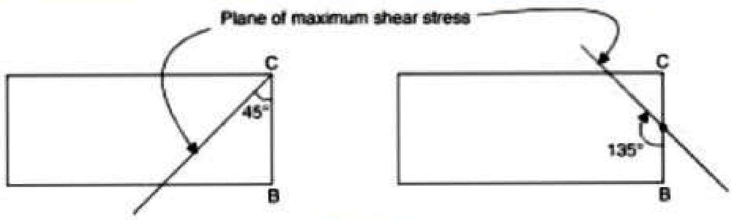

Fig. 3.4 (*c*)

Hence the planes, which are at an angle of 45° or 135° with the normal cross-section *BC* [see Fig. 3.4 (c)], carry the maximum shear stresses.

Principal planes. Principal planes are the planes on which shear stress is zero. To locate the position of principal planes, the shear stress given by equation (3.7) should be equated to zero.

$\therefore$ For principal planes,

$$\frac{\sigma_1 - \sigma_2}{2} \sin 2\theta = 0$$

or $\sin 2\theta = 0$ [$\because$ $(\sigma_1 - \sigma_2)$ cannot be equal to zero]

or $2\theta = 0$ or $180°$

$\therefore$ $\theta = 0$ or $90°$

when $\theta = 0$,

$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 0°$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \times 1 \quad (\because \cos 0° = 1)$$

$$= \sigma_1$$

when $\theta = 90°$,

$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2 \times 90°$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 180°$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \times (-1) \quad (\because \cos 180° = -1)$$

$$= \sigma_2.$$

Note. The relations, given by equations (3.6) to (3.9), also hold good when one or both the stresses are compressive.

Problem 3.5. *The tensile stresses at a point across two mutually perpendicular planes are 120 N/mm² and 60 N/mm². Determine the normal, tangential and resultant stresses on a plane inclined at 30° to the axis of the minor stress.*

Sol. Given :

Major principal stress, $\sigma_1 = 120$ N/mm²

Minor principal, $\sigma_2 = 60$ N/mm²

Angle of oblique plane with the axis of minor principal stress,

$\theta = 30°$.

Normal stress

The normal stress (σ_n) is given by equation (3.6),

$$\therefore \quad \sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta$$

$$= \frac{120 + 60}{2} + \frac{120 - 60}{2} \cos 2 \times 30°$$

$$= 90 + 30 \cos 60° = 90 + 30 \times \frac{1}{2}$$

$$= \mathbf{105\ N/mm^2}.$$ **Ans.**

Tangential stress

The tangential (or shear stress) σ_t is given by equation (3.7).

$$\therefore \quad \sigma_t = \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta$$

$$= \frac{120 - 60}{2} \sin (2 \times 30^\circ)$$

$$= 30 \times \sin 60^\circ = 30 \times 0.866$$

$$= \mathbf{25.98\ N/mm^2}. \quad \textbf{Ans.}$$

Resultant stress

The resultant stress (σ_R) is given by equation (3.8)

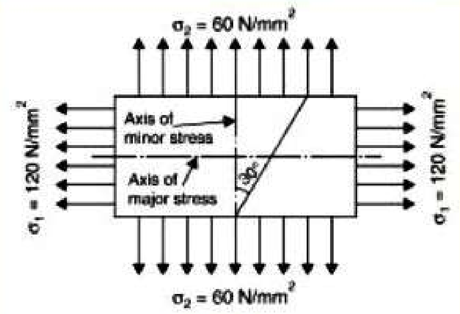

Fig. 3.5

$$\therefore \quad \sigma_R = \sqrt{\sigma_n^2 + \sigma_t^2} = \sqrt{105^2 + 25.98^2}$$

$$= \sqrt{11025 + 674.96} = \mathbf{108.16\ N/mm^2}. \quad \textbf{Ans.}$$

Problem 3.6. *The stresses at a point in a bar are 200 N/mm² (tensile) and 100 N/mm² (compressive). Determine the resultant stress in magnitude and direction on a plane inclined at 60° to the axis of the major stress. Also determine the maximum intensity of shear stress in the material at the point.*

Sol. Given :

Major principal stress, $\sigma_1 = 200\ N/mm^2$

Minor principal stress, $\sigma_2 = -100\ N/mm^2$

(Minus sign is due to compressive stress)

Angle of the plane, which it makes with the major principal stress = 60°

$\therefore$ Angle $\theta = 90^\circ - 60^\circ = 30^\circ$.

Resultant stress in magnitude and direction

First calculate the normal and tangential stresses.

Using equation (3.6) for normal stress,

$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta$$

$$= \frac{200 + (-100)}{2} + \frac{200 - (-100)}{2} \cos (2 \times 30^\circ) \quad (\because \theta = 30^\circ)$$

$$= \frac{200 - 100}{2} + \frac{200 + 100}{2} \cos 60^\circ$$

$$= 50 + 150 \times \tfrac{1}{2} \quad (\because \cos 60^\circ = \tfrac{1}{2})$$

$$= 50 + 75 = 125\ N/mm^2.$$

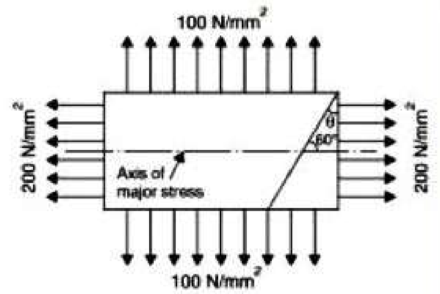

Fig. 3.6

Using equation (3.7) for tangential stress,

$$\sigma_t = \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta = \frac{200 - (-100)}{2} \sin (2 \times 30^\circ)$$

$$= \frac{200 + 100}{2} \sin 60^\circ = 150 \times 0.866 = 129.9\ N/mm^2.$$

Using equation (3.8) for resultant stress,

$$\sigma_R = \sqrt{\sigma_n^2 + \sigma_t^2} = \sqrt{125^2 + 129.9^2}$$

$$= \sqrt{15625 + 16874} = \mathbf{180.27\ N/mm^2}. \quad \textbf{Ans.}$$

The inclination of the resultant stress with the normal of the inclined plane is given by equation [3.8 (A)] as

$$\tan \phi = \frac{\sigma_t}{\sigma_n} = \frac{129.9}{125} = 1.04$$

$$\therefore \quad \phi = \tan^{-1} 1.04 = \mathbf{46^\circ\ 6'}. \quad \textbf{Ans.}$$

Maximum shear stress

Maximum shear stress is given by equation (3.9)

$$\therefore \quad (\sigma_t)_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{200-(-100)}{2} = \frac{200+100}{2} = \mathbf{150\ N/mm^2.}\quad \textbf{Ans.}$$

Problem 3.8. *At a point in a strained material the principal stresses are 100 N/mm² (tensile) and 60 N/mm² (compressive). Determine the normal stress, shear stress and resultant stress on a plane inclined at 50° to the axis of major principal stress. Also determine the maximum shear stress at the point.*

Sol. Given :

Major principal stress, $\sigma_1 = 100\ N/mm^2$

Minor principal stress, $\sigma_2 = -60\ N/mm^2$ (Negative sign due to compressive stress)

Angle of the inclined plane with the axis of major principal stress = 50°

∴ Angle of the inclined plane with the axis of minor principal stress,

$$\theta = 90 - 50 = 40°.$$

Normal stress (σ_n)

Using equation (3.6),

$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2}\cos 2\theta$$

$$= \frac{100+(-60)}{2} + \frac{100-(-60)}{2}\cos(2\times 40°)$$

$$= \frac{100-60}{2} + \frac{100+60}{2}\cos 80°$$

$$= 20 + 80 \times \cos 80° = 20 + 80 \times .1736$$

$$= 20 + 13.89 = \mathbf{33.89\ N/mm^2.}\quad \textbf{Ans.}$$

Shear stress (σ_t)

Using equation (3.7), $\sigma_t = \frac{\sigma_1 - \sigma_2}{2}\sin 2\theta$

$$= \frac{100-(-60)}{2}\sin(2\times 40°)$$

$$= \frac{100+60}{2}\sin 80° = 80 \times 0.9848 = \mathbf{78.785\ N/mm^2.}\quad \textbf{Ans.}$$

Resultant stress (σ_R)

Using equation on (3.8),

$$\sigma_R = \sqrt{\sigma_n^2 + \sigma_t^2} = \sqrt{33.89^2 + 78.785^2}$$

$$= \sqrt{1148.53 + 6207.07} = \mathbf{85.765\ N/mm^2.}\quad \textbf{Ans.}$$

Maximum shear stress

Using equation (3.9),

$$(\sigma_t)_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{100-(-60)}{2}$$

$$= \frac{100+60}{2} = \mathbf{80\ N/mm^2.}\quad \textbf{Ans.}$$

Problem 3.9. *At a point in a strained material, the principal stresses are 100 N/mm² tensile and 40 N/mm² compressive. Determine the resultant stress in magnitude and direction on a plane inclined at 60° to the axis of the major principal stress. What is the maximum intensity of shear stress in the material at the point ?*

Sol. Given :

The major principal stress, $\sigma_1 = 100$ N/mm²

The minor principal stress, $\sigma_2 = -40$ N/mm² (Minus sign due to compressive stress)

Inclination of the plane with the axis of major principal stress = 60°

∴ Inclination of the plane with the axis of minor principal stress,

$$\theta = 90 - 60 = 30°.$$

Resultant stress in magnitude

The resultant stress (σ_R) is given by equation (3.8) as

$$\sigma_R = \sqrt{\sigma_n^2 + \sigma_t^2}$$

where σ_n = Normal stress and is given by equation (3.6) as

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta$$

$$= \frac{100 + (-40)}{2} + \frac{100 - (-40)}{2} \cos (2 \times 30°)$$

$$= \frac{100 - 40}{2} + \frac{100 + 40}{2} \cos 60°$$

$$= 30 + 70 \times 0.5 \qquad (\because \cos 60° = 0.5)$$

$$= 65 \text{ N/mm}^2$$

and σ_t = Shear stress and is given by equation (3.7) as

$$= \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta = \frac{100 - (-40)}{2} \sin (2 \times 30°)$$

$$= \frac{100 + 40}{2} \sin 60° = 70 \times .866 = 60.62 \text{ N/mm}^2$$

$$\sigma_R = \sqrt{65^2 + 60.62^2} = \mathbf{88.9 \text{ N/mm}^2}. \quad \textbf{Ans.}$$

Direction of resultant stress

Let the resultant stress is inclined at an angle ϕ to the normal of the oblique plane. Then using equation [3.8 (*A*)].

$$\tan \phi = \frac{\sigma_t}{\sigma_n} = \frac{60.62}{65}$$

$$\therefore \qquad \phi = \tan^{-1} \frac{60.62}{65} = \mathbf{43°}. \quad \textbf{Ans.}$$

Maximum shear stress

Using equation (3.9), $(\sigma_t)_{max} = \dfrac{\sigma_1 - \sigma_2}{2}$

$$= \frac{100 - (-40)}{2} = \frac{100 + 40}{2} = \mathbf{70 \text{ N/mm}^2}. \quad \textbf{Ans.}$$

3.4.3. A Member Subjected to a Simple Shear Stress.

Fig. 3.8 shows a rectangular bar *ABCD* of uniform cross-sectional area *A* and of unit thickness. The bar is subjected to a simple shear stress (q) across the faces *BC* and *AD*. Let *FC* be the oblique section on which normal and tangential stresses are to be calculated.

Let θ = Angle made by oblique section *FC* with normal cross-section *BC*,

τ = Shear stress across faces *BC* and *AD*.

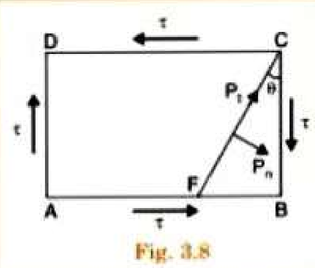

Fig. 3.8

It has already been proved (Refer to Art. 2.9) that a shear stress is always accompanied by an equal shear stress at right angles to it. Hence the faces *AB* and *CD* will also be subjected to a shear stress q as shown in Fig. 3.8. Now these stresses will be converted into equivalent forces. Then these forces will be resolved along the inclined surface and normal to inclined surface. Consider the forces acting on the wedge *FBC* of Fig. 3.9.

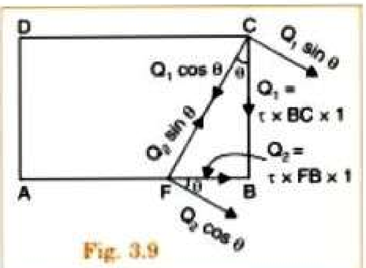

Fig. 3.9

Let Q_1 = Shear force on face *BC*

= Shear stress × Area of face *BC*

$= \tau \times BC \times 1$ (∵ Area of face $BC = BC \times 1$)

$= \tau \times BC$

Q_2 = Shear force on face *FB*

$= \tau \times$ Area of *FB*

$= \tau \times FB \times 1 = \tau . FB$

P_n = Total normal force on section *FC*

P_t = Total tangential force on section *FC*.

The force Q_1 is acting along face *CB* as shown in Fig. 3.9. This force is resolved into two components *i.e.*, $Q_1 \cos\theta$ and $Q_1 \sin\theta$ along the plane *CF* and normal to the plane *CF* respectively.

The force Q_2 is acting along the face *FB*. This force is also resolved into two components *i.e.*, $Q_2 \sin\theta$ and $Q_2 \cos\theta$ along the plane *FC* and normal to the plane *FC* respectively.

∴ Total normal force on section *FC*,

$$P_n = Q_1 \sin\theta + Q_2 \cos\theta$$

$$= \tau \times BC \times \sin\theta + \tau \times FB \times \cos\theta. \quad (\because\ Q_1 = \tau \times BC \text{ and } Q_2 = \tau \times FB)$$

And total tangential force on section *FC*.

$$P_t = Q_2 \sin\theta - Q_1 \cos\theta. \quad (\text{–ve sign is taken due to opposite direction})$$

$$= \tau \times FB \times \sin\theta - \tau \times BC \times \cos\theta \quad (\because\ Q_2 = \tau . FB \text{ and } Q_1 = \tau . BC)$$

Let σ_n = Normal stress on section *FC*

σ_t = Tangential stress on section *FC*

Then
$$\sigma_n = \frac{\text{Total normal force on section } FC}{\text{Area of section } FC}$$

$$= \frac{P_n}{FC \times 1}$$

$$= \frac{\tau . BC . \sin\theta + \tau . FB . \cos\theta}{FC \times 1} \quad (\because\ \text{Area} = FC \times 1)$$

$$= \tau . \frac{BC}{FC} . \sin\theta + \tau . \frac{FB}{FC} . \cos\theta$$

$$= \tau . \cos\theta . \sin\theta + \tau . \sin\theta . \cos\theta$$

$$\left(\because\ \text{In triangle } FBC, \frac{BC}{FC} = \cos\theta, \frac{FB}{FC} = \sin\theta\right)$$

$$= 2\tau \cos\theta . \sin\theta$$

$$= \tau \sin 2\theta \quad (\because\ 2 \sin\theta \cos\theta = \sin 2\theta) \quad \ldots(3.10)$$

and
$$\sigma_t = \frac{\text{Total tangential force on section } FC}{\text{Area of section } FC}$$

$$= \frac{P_t}{FC \times 1}$$

$$= \frac{\tau \times FB \times \sin\theta - \tau \times BC \times \cos\theta}{FC \times 1}$$

$= \tau \times \frac{FB}{FC} \times \sin\theta - \tau \times \frac{BC}{FC} \times \cos\theta$
$= \tau \times \sin\theta \times \sin\theta - \tau \times \cos\theta \times \cos\theta$

$= \tau \sin^2\theta - \tau\cos^2\theta = -\tau[\cos^2\theta - \sin^2\theta]$
$= -\tau\cos 2\theta$ $\quad (\because \cos^2\theta - \sin^2\theta = \cos 2\theta)$ ...(3.11)

–ve sign shows that σ_t will be acting downwards on the plane *CF*.

3.4.4. A Member Subjected to Direct Stresses in two Mutually Perpendicular Directions Accompanied by a Simple Shear Stress. Fig. 3.10 (*a*) shows a rectangular bar *ABCD* of uniform cross-sectional area *A* and of unit thickness. This bar is subjected to :

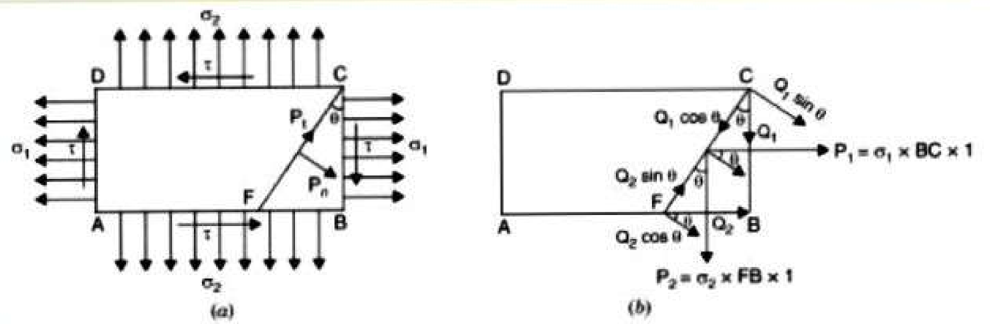

Fig. 3.10

(*i*) tensile stress σ_1 on the face *BC* and *AD*

(*ii*) tensile stress σ_2 on the face *AB* and *CD*

(*iii*) a simple shear stress τ on face *BC* and *AD*.

But with reference to Art. 2.9, a simple shear stress is always accompanied by an equal shear stress at right angles to it. Hence the faces *AB* and *CD* will also be subjected to a shear stress τ as shown in Fig. 3.10 (*a*).

We want to calculate normal and tangential stresses on oblique section *FC*, which is inclined at an angle θ with the normal cross-section *BC*. The given stresses are converted into equivalent forces.

The forces acting on the wedge *FBC* are :

P_1 = Tensile force on face *BC* due to tensile stress σ_1
$= \sigma_1 \times$ Area of *BC*
$= \sigma_1 \times BC \times 1$ $\quad (\because$ Area $= BC \times 1)$
$= \sigma_1 \times BC$

P_2 = Tensile force on face *FB* due to tensile stress σ_2
$= \sigma_2 \times$ Area of $FB = \sigma_2 \times FB \times 1$
$= \sigma_2 \times FB$

Q_1 = Shear force on face *BC* due to shear stress τ
$= \tau \times$ Area of *BC*

$= \tau \times BC \times 1 = \tau \times BC$

Q_2 = Shear force on face *FB* due to shear stress τ
$= \tau \times$ Area of *FB*
$= \tau \times FB \times 1 = \tau \times FB.$

Resolving the above four forces (*i.e.*, P_1, P_2, Q_1 and Q_2) normal to the oblique section *FC*, we get

Total normal force,

$$P_n = P_1\cos\theta + P_2\sin\theta + Q_1\sin\theta + Q_2\cos\theta$$

Substituting the values of P_1, P_2, Q_1 and Q_2, we get

$$P_n = \sigma_1 . BC . \cos\theta + \sigma_2 . FB . \sin\theta + \tau . BC . \sin\theta + \tau . FB . \cos\theta$$

Similarly, the total tangential force (P_t) is obtained by resolving P_1, P_2, Q_1 and Q_2 along the oblique section *FC*.

∴ Total tangential force,

$$P_t = P_1 \sin\theta - P_2 \cos\theta - Q_1 \cos\theta + Q_2 \sin\theta$$
$$= \sigma_1 . BC . \sin\theta - \sigma_2 . FB . \cos\theta - \tau . BC . \cos\theta + \tau . FB . \sin\theta$$

(substitute the values of P_1, P_2, Q_1 and Q_2)

Now, Let σ_n = Normal stress across the section FC, and
σ_t = Tangential stress across the section FC.

Then normal stress across the section FC,

$$\sigma_n = \frac{\text{Total normal force across section } FC}{\text{Area of section } FC} = \frac{P_n}{FC \times 1}$$
$$= \frac{\sigma_1 . BC . \cos\theta + \sigma_2 . FB . \sin\theta + \tau . BC . \sin\theta + \tau . FB . \cos\theta}{FC \times 1}$$
$$= \sigma_1 . \frac{BC}{FC} . \cos\theta + \sigma_2 . \frac{FB}{FC} . \sin\theta + \tau . \frac{BC}{FC} . \sin\theta + \tau . \frac{FB}{FC} . \cos\theta$$
$$= \sigma_1 . \cos\theta . \cos\theta + \sigma_2 \sin\theta . \sin\theta + \tau . \cos\theta . \sin\theta + \tau \sin\theta . \cos\theta$$

$$\left(\because \text{ In triangle } FBC, \frac{BC}{FC} = \cos\theta \text{ and } \frac{FB}{FC} = \sin\theta\right)$$

$$= \sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta + 2\tau \cos\theta \sin\theta$$
$$= \sigma_1 \left(\frac{1+\cos 2\theta}{2}\right) + \sigma_2 \left(\frac{1-\cos 2\theta}{2}\right) + \tau \sin 2\theta$$

$$\left(\because \cos^2\theta = \frac{1+\cos 2\theta}{2}, \sin^2\theta = \frac{1-\cos 2\theta}{2} \text{ and } 2\cos\theta\sin\theta = \sin 2\theta\right)$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta + \tau \sin 2\theta \quad \text{...(3.12)}$$

d tangential stress (*i.e.*, shear stress) across the section FC,

$$\sigma_t = \frac{\text{Total tangential force across section } FC}{\text{Area of section } FC} = \frac{P_t}{FC \times 1}$$
$$= \frac{\sigma_1 . BC . \sin\theta - \sigma_2 . FB . \cos\theta - \tau . BC . \cos\theta + \tau . FB . \sin\theta}{FC \times 1}$$
$$= \sigma_1 . \frac{BC}{FC} . \sin\theta - \sigma_2 . \frac{FB}{FC} . \cos\theta - \tau . \frac{BC}{FC} . \cos\theta + \tau . \frac{FB}{FC} . \sin\theta$$
$$= \sigma_1 . \cos\theta . \sin\theta - \sigma_2 . \sin\theta . \cos\theta - \tau . \cos\theta . \cos\theta + \tau . \sin\theta . \sin\theta$$

$$\left(\because \text{ In triangle } FBC, \frac{BC}{FC} = \cos\theta \text{ and } \frac{FB}{FC} = \sin\theta\right)$$

$$= (\sigma_1 - \sigma_2) . \cos\theta \sin\theta - \tau \cos^2\theta + \tau \sin^2\theta$$
$$= \left(\frac{\sigma_1 - \sigma_2}{2}\right) . 2\cos\theta\sin\theta - \tau(\cos^2\theta - \sin^2\theta)$$

$$= \frac{\sigma_1 - \sigma_2}{2} . \sin 2\theta - \tau \cos 2\theta \qquad (\because \cos^2\theta - \sin^2\theta = \cos 2\theta) \quad \text{...(3.13)}$$

Position of principal planes. The planes on which shear stress (*i.e.*, tangential stress) is zero, are known as principal planes. And the stresses acting on principal planes are known as principal stresses.

The position of principal planes are obtained by equating the tangential stress [given by equation (3.13)] to zero.

∴ For principal planes, $\sigma_t = 0$

or $$\frac{\sigma_1 - \sigma_2}{2} \sin 2\theta - \tau \cos 2\theta = 0$$

or $$\frac{\sigma_1 - \sigma_2}{2} \sin 2\theta = \tau \cos 2\theta$$

or $$\frac{\sin 2\theta}{\cos 2\theta} = \frac{\tau}{\frac{(\sigma_1 - \sigma_2)}{2}} = \frac{2\tau}{(\sigma_1 - \sigma_2)}$$

or $$\tan 2\theta = \frac{2\tau}{(\sigma_1 - \sigma_2)} \quad \text{...(3.14)}$$

But the tangent of any angle in a right angled triangle

$$= \frac{\text{Height of right angled triangle}}{\text{Base of right angled triangle}}$$

$$\therefore \quad \frac{\text{Height of right angled triangle}}{\text{Base of right angled triangle}} = \frac{2\tau}{(\sigma_1 - \sigma_2)}$$

$\therefore$ Height of right angled triangle = 2τ

Base of right angled triangle = $(\sigma_1 - \sigma_2)$.

Now diagonal of the right angled triangle

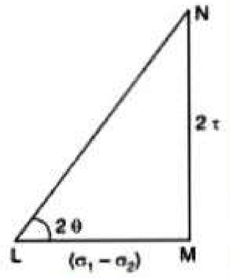

Fig. 3.11

$$= \pm\sqrt{(\sigma_1 - \sigma_2)^2 + (2\tau)^2} = \pm\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}$$

$$= \sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2} \quad \text{and} \quad -\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}$$

1st Case. Diagonal $= \sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}$

Then $$\sin 2\theta = \frac{\text{Height}}{\text{Diagonal}} = \frac{2\tau}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

and $$\cos 2\theta = \frac{\text{Base}}{\text{Diagonal}} = \frac{(\sigma_1 - \sigma_2)}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}.$$

The value of major principal stress is obtained by substituting the values of sin 2θ and cos 2θ in equation (3.12).

$\therefore$ Major principal stress

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2}\cos 2\theta + \tau \sin 2\theta$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \times \frac{(\sigma_1 - \sigma_2)}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}} + \tau \times \frac{2\tau}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{1}{2}\frac{(\sigma_1 - \sigma_2)^2}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}} + \frac{2\tau^2}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{(\sigma_1 - \sigma_2)^2 + 4\tau^2}{2\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{1}{2}\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2} \qquad \text{...(3.15)}$$

2nd Case. Diagonal $= -\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}$

Then $$\sin 2\theta = \frac{2\tau}{-\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

and $$\cos 2\theta = \frac{(\sigma_1 - \sigma_2)}{-\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

Substituting these values in equation (3.12), we get minor principal stress.

$\therefore$ Minor principal stress

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2}\cos 2\theta + \tau \sin 2\theta$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \times \frac{\sigma_1 - \sigma_2}{-\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}} + \tau \times \frac{2\tau}{-\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

$$= \frac{\sigma_1 + \sigma_2}{2} - \frac{(\sigma_1 - \sigma_2)^2}{2\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}} - \frac{2\tau^2}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

$$= \frac{\sigma_1 + \sigma_2}{2} - \frac{(\sigma_1 - \sigma_2)^2 + 4\tau^2}{2\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

$$= \frac{\sigma_1 + \sigma_2}{2} - \frac{1}{2}\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}$$

$$= \frac{\sigma_1 + \sigma_2}{2} - \sqrt{\left(\frac{(\sigma_1 - \sigma_2)}{2}\right)^2 + \tau^2} \qquad ...(3.16)$$

Equation (3.15) gives the maximum principal stress whereas equation (3.16) gives minimum principal stress. These two principal planes are at right angles.

The position of principal planes is obtained by finding two values of θ from equation (3.14).

Fig. 3.11 (*a*) shows the principal planes in which θ_1 and θ_2 are the values from equation (3.14).

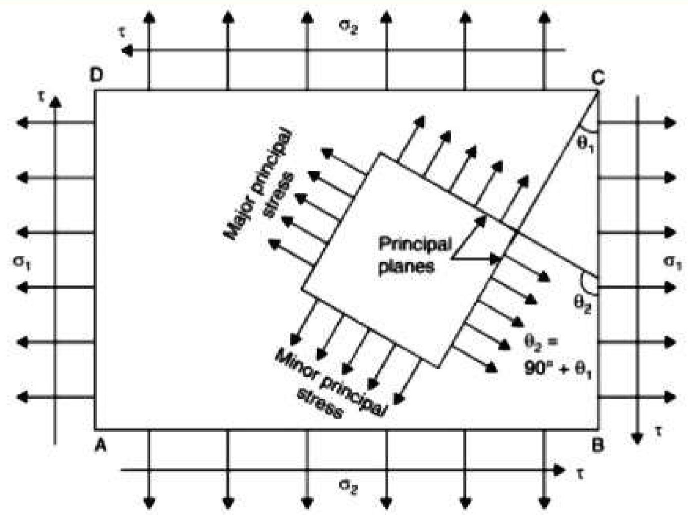

Fig. 3.11 (*a*)

Maximum shear stress. The shear stress is given by equation (3.13). The shear stress will be maximum or minimum when

$$\frac{d}{d\theta}(\sigma_t) = 0$$

or $$\frac{d}{d\theta}\left[\frac{\sigma_1 - \sigma_2}{2}\sin 2\theta - \tau\cos 2\theta\right] = 0$$

or $$\frac{\sigma_1 - \sigma_2}{2}(\cos 2\theta) \times 2 - \tau(-\sin 2\theta) \times 2 = 0$$

$$(\sigma_1 - \sigma_2) \,.\, \cos 2\theta + 2\tau \sin 2\theta = 0$$

or $$2\tau \sin 2\theta = -(\sigma_1 - \sigma_2)\cos 2\theta$$

$$= (\sigma_2 - \sigma_1)\cos 2\theta$$

or $$\frac{\sin 2\theta}{\cos 2\theta} = \frac{\sigma_2 - \sigma_1}{2\tau}$$

or $$\tan 2\theta = \frac{\sigma_2 - \sigma_1}{2\tau} \qquad ...(3.17)$$

Equation (3.17) gives condition for maximum or minimum shear stress.

If $\tan 2\theta = \frac{\sigma_2 - \sigma_1}{2\tau}$

Then $$\sin 2\theta = \pm \frac{\sigma_2 - \sigma_1}{\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}}$$

and $$\cos 2\theta = \pm \frac{2\tau}{\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}}$$

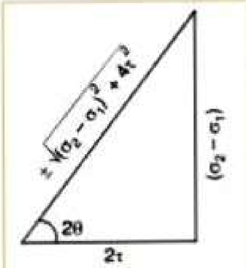

Fig. 3.12

Substituting the values of sin 2θ and cos 2θ in equation (3.13), the maximum and minimum shear stresses are obtained.

∴ Maximum shear stress is given by

$$(\sigma_t)_{max} = \frac{\sigma_1 - \sigma_2}{2}\sin 2\theta - \tau\cos 2\theta$$

$$= \pm \frac{\sigma_1 - \sigma_2}{2} \times \frac{(\sigma_2 - \sigma_1)}{\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}} \pm \tau \times \frac{2\tau^2}{\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}}$$

$$= \pm \frac{(\sigma_1 - \sigma_2)^2}{2\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}} \pm \frac{2\tau^2}{\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}}$$

$$= \pm \frac{(\sigma_2 - \sigma_1)^2 + 4\tau^2}{2\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}} = \pm \frac{1}{2}\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}$$

$$\therefore \quad (\sigma_t)_{max} = \frac{1}{2}\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}$$

$$= \frac{1}{2}\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2} \quad \text{...(3.18)}$$

The planes on which maximum shear stress is acting, are obtained after finding the two values of θ from equation (3.17). These two values of θ will differ by 90°.

The second method of finding the planes of maximum shear stress is to find first principal planes and principal stresses. Let θ_1 is the angle of principal plane with plane *BC* of Fig. 3.11 (*a*). Then the planes of maximum shear will be at $\theta_1 + 45°$ and $\theta_1 + 135°$ with the plane *BC* as shown in Fig. 3.12 (*a*).

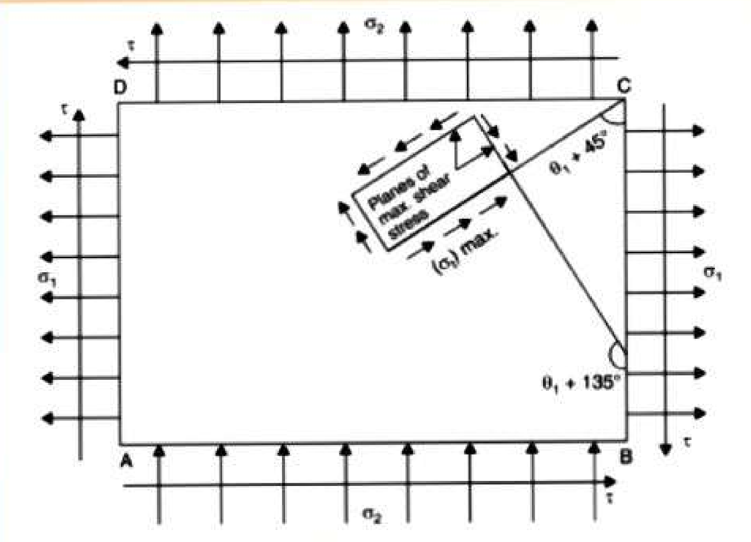

Fig. 3.12 (*a*)

Note. The above relations hold good when one or both the stresses are compressive.

Problem 3.11. *At a point within a body subjected to two mutually perpendicular directions, the stresses are 80 N/mm² tensile and 40 N/mm² tensile. Each of the above stresses is accompanied by a shear stress of 60 N/mm². Determine the normal stress, shear stress and resultant stress on an oblique plane inclined at an angle of 45° with the axis of minor tensile stress.*

Sol. Given :

Major tensile stress, $\sigma_1 = 80$ N/mm²

Minor tensile stress, $\sigma_2 = 40$ N/mm²

Shear stress, $\tau = 60$ N/mm²

Angle of oblique plane, with the axis of minor tensile stress,

$\theta = 45°$.

(i) Normal stress (σ_n)

Using equation (3.12),

$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2}\cos 2\theta + \tau \sin 2\theta$$

$$= \frac{80+40}{2} + \frac{80-40}{2}\cos(2 \times 45°) + 60 \sin(2 \times 45°)$$

$$= 60 + 20 \cos 90° + 60 \sin 90°$$

$$= 60 + 20 \times 0 + 60 \times 1 \qquad (\because \cos 90° = 0)$$

$$= 60 + 0 + 60 = \mathbf{120\ N/mm^2}.$$ **Ans.**

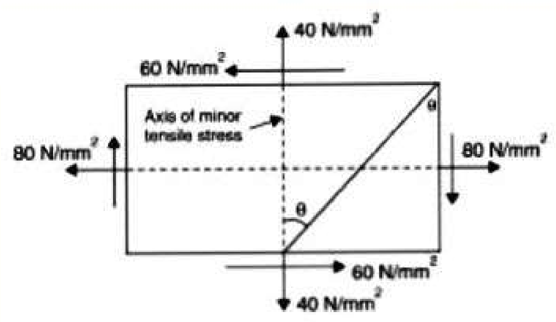

Fig. 3.13

(ii) Shear (or tangential) stress (σ_t)

Using equation (3.13),

$$\sigma_t = \frac{\sigma_1 - \sigma_2}{2}\sin 2\theta - \tau \cos 2\theta$$

$$= \frac{80-40}{2}\sin(2 \times 45°) - 60 \times \cos(2 \times 45°)$$

$$= 20 \times \sin 90° - 60 \cos 90°$$

$$= 20 \times 1 - 60 \times 0$$

$$= \mathbf{20\ N/mm^2}.$$ **Ans.**

(iii) Resultant stress (σ_R)

Using equation, $\sigma_R = \sqrt{\sigma_n^2 + \sigma_t^2}$

$$= \sqrt{120^2 + 20^2} = \sqrt{14400 + 400}$$

$$= \sqrt{14800} = \mathbf{121.655\ N/mm^2}.$$ **Ans.**

Problem 3.12. *A rectangular block of material is subjected to a tensile stress of 110 N/mm² on one plane and a tensile stress of 47 N/mm² on the plane at right angles to the former. Each of the above stresses is accompanied by a shear stress of 63 N/mm² and that associated with the former tensile stress tends to rotate the block anticlockwise. Find :*

(i) the direction and magnitude of each of the principal stress and

(ii) magnitude of the greatest shear stress.

Sol. Given :

Major tensile stress, $\sigma_1 = 110\ \text{N/mm}^2$

Minor tensile stress, $\sigma_2 = 47\ \text{N/mm}^2$

Shear stress, $\tau = 63\ \text{N/mm}^2$

(*i*) Major principal stress is given by equation (3.15).

$$\therefore \text{ Major principal stress} = \frac{\sigma_1+\sigma_2}{2} + \sqrt{\left(\frac{\sigma_1-\sigma_2}{2}\right)^2 + \tau^2}$$

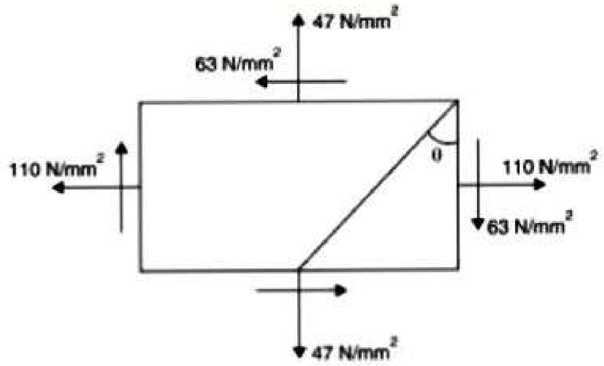

Fig. 3.14

$$= \frac{110+47}{2} + \sqrt{\left(\frac{110-47}{2}\right)^2 + 63^2}$$

$$= \frac{157}{2} + \sqrt{\left(\frac{63}{2}\right)^2 + (63)^2}$$

$$= 78.5 + \sqrt{31.5^2 + 63^2} = 78.5 + \sqrt{992.25 + 3969}$$

$$= 78.5 + 70.436 = \mathbf{148.936\ N/mm^2}.$$ **Ans.**

Minor principal stress is given by equation (3.16).

$$\therefore \text{ Minor principal stress}, \quad = \frac{\sigma_1+\sigma_2}{2} - \sqrt{\left(\frac{\sigma_1-\sigma_2}{2}\right)^2 + \tau^2}$$

$$= \frac{110+47}{2} - \sqrt{\left(\frac{110-47}{2}\right)^2 + 63^2} = 78.5 - 70.436$$

$$= \mathbf{8.064\ N/mm^2}.$$ **Ans.**

The directions of principal stresses are given by equation (3.14).

$\therefore$ Using equation (3.14),

$$\tan 2\theta = \frac{2\tau}{\sigma_1-\sigma_2} = \frac{2\times 63}{110-47}$$

$$= \frac{2\times 63}{63} = 2.0$$

$\therefore$ $2\theta = \tan^{-1} 2.0 = 63^\circ 26'$ or $243^\circ 26'$

$\therefore$ $\theta = \mathbf{31^\circ 43'}$ **or** $\mathbf{121^\circ 43'}$. **Ans.**

(*ii*) *Magnitude of the greatest shear stress*

Greatest shear stress is given by equation (3.18).

Using equation (3.18),

$$(\sigma_t)_{max} = \frac{1}{2}\sqrt{(\sigma_1-\sigma_2)^2 + 4\tau^2}$$

$$= \frac{1}{2}\sqrt{(100-47)^2 + 4\times 63^2}$$

$$= \frac{1}{2}\sqrt{63^2 + 4 \times 63^2} = \frac{1}{2} \times 63 \times \sqrt{5}$$

$= 70.436$ **N/mm²**. **Ans.**

3.5. MOHR'S CIRCLE

Mohr's circle is a graphical method of finding normal, tangential and resultant stresses on an oblique plane. Mohr's circle will be drawn for the following cases :

(*i*) A body subjected to two mutually perpendicular principal tensile stresses of unequal intensities.

(*ii*) A body subjected to two mutually perpendicular principal stresses which are unequal and unlike (*i.e.*, one is tensile and other is compressive).

(*iii*) A body subjected to two mutually perpendicular principal tensile stresses accompanied by a simple shear stress.

3.5.1. Mohr's Circle when a Body is Subjected to two Mutually Perpendicular Principal Tensile Stresses of Unequal Intensities. Consider a rectangular body subjected to two mutually perpendicular principal tensile stresses of unequal intensities. It is required to find the resultant stress on an oblique plane.

Let σ_1 = Major tensile stress

σ_2 = Minor tensile stress, and

θ = Angle made by the oblique plane with the axis of minor tensile stress.

Mohr's circle is drawn as : (See Fig. 3.18).

Take any point *A* and draw a horizontal line through *A*. Take $AB = \sigma_1$ and $AC = \sigma_2$ towards right from *A* to some suitable scale. With *BC* as diameter describe a circle. Let *O* is the centre of the circle. Now through *O*, draw a line *OE* making an angle 2θ with *OB*.

From *E*, draw *ED* perpendicular on *AB*. Join *AE*. Then the normal and tangential stresses on the oblique plane are given by *AD* and *ED* respectively. The resultant stress on the oblique plane is given by *AE*.

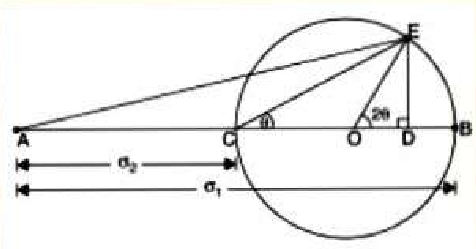

Fig. 3.18

From Fig. 3.18, we have

Length *AD* = Normal stress on oblique plane

Length *ED* = Tangential stress on oblique plane

Length *AE* = Resultant stress on oblique plane.

Radius of Mohr's circle $= \frac{\sigma_1 - \sigma_2}{2}$

Angle ϕ = obliquity.

Proof. (See Fig. 3.18)

$$CO = OB = OE = \text{Radius of Mohr's circle} = \frac{\sigma_1 - \sigma_2}{2}$$

$$\therefore \quad AO = AC + CO$$

$$= \sigma_2 + \frac{\sigma_1 - \sigma_2}{2} = \frac{2\sigma_2 + \sigma_1 - \sigma_2}{2} = \frac{\sigma_1 + \sigma_2}{2}$$

$$OD = OE \cos 2\theta$$

$$= \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta \qquad \left(\because \ OE = \frac{\sigma_1 - \sigma_2}{2}\right)$$

$$\therefore \quad AD = AO + OD$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta$$

$= \sigma_n$ or Normal stress

and $$ED = OE \sin 2\theta$$

$$= \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta$$

$= \sigma_t$ or Tangential stress.

(i) Normal stress is along the line *ACB*. Hence maximum normal stress will be when point *E* is at *B*. And minimum normal stress will be when point *E* is at *C*. Hence maximum normal stress = $AB = \sigma_1$ and minimum normal stress = $AB = \sigma_2$.

(ii) Tangential stress (or shear stress) is along a line which is perpendicular to line *CB*. Hence maximum shear stress will be when perpendicular to line *CB* is drawn from point *O*. Then maximum shear stress will be equal to the radius of the Mohr's circle.

$$\therefore \qquad (\sigma_t)_{max} = \frac{\sigma_1 - \sigma_2}{2}.$$

(iii) When the point *E* is at *B* or at *C*, the shear stress will be zero.

(iv) The angle ϕ (which is known as angle of obliquity) will be maximum, when the line *AE* is tangent to the Mohr's circle.

3.5.2. Mohr's Circle when a Body is Subjected to two Mutually Perpendicular Principal Stresses which are Unequal and Unlike (i.e., one is Tensile and other is Compressive). Consider a rectangular body subjected to two mutually perpendicular principal stresses which are unequal and one of them is tensile and the other is compressive. It is required to find the resultant stress on an oblique plane.

Let σ_1 = Major principal tensile stress,

σ_2 = Minor principal compressive stress, and

θ = Angle made by the oblique plane with the axis of minor principal stress.

Mohr's circle is drawn as : (See Fig. 3.20)

Take any point *A* and draw a horizontal line through *A* on both sides of *A* as shown in Fig. 3.20. Take $AB = \sigma_1(+)$ towards right of *A* and $AC = \sigma_2(-)$ towards left of *A* to some suitable scale. Bisect *BC* at *O*. With *O* as centre and radius equal to *CO* or *OB*, draw a circle. Through *O* draw a line *OE* making an angle 2θ with *OB*.

From *E*, draw *ED* perpendicular to *AB*. Join *AE* and *CE*. Then normal and shear stress (*i.e.*, tangential stress) on the oblique plane are given by *AD* and *ED*. Length *AE* represents the resultant stress on the oblique plane.

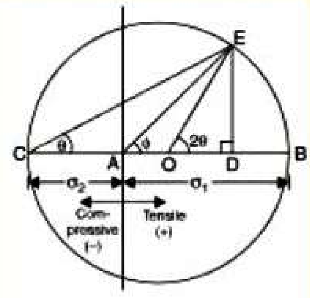

Fig. 3.20

∴ From Fig. 3.20, we have

Length *AD* = Normal stress on oblique plane,

Length *ED* = Shear stress on oblique plane,

Length *AE* = Resultant stress on oblique plane, and

Angle ϕ = Obliquity.

Radius of Mohr's circle = *CO* or $OB = \dfrac{\sigma_1 + \sigma_2}{2}$.

Proof. (See Fig. 3.20).

$$CO = OB = OE = \text{Radius of Mohr's circle} = \frac{\sigma_1 + \sigma_2}{2}$$

$$AO = OC - AC = \frac{\sigma_1 + \sigma_2}{2} - \sigma_2 = \frac{\sigma_1 + \sigma_2 - 2\sigma_2}{2} = \frac{\sigma_1 - \sigma_2}{2}$$

$$\therefore \quad AD = AO + OD$$

$$= AO + OB \cos 2\theta \qquad (\because OD = OE \cos 2\theta)$$

$$= \frac{\sigma_1 - \sigma_2}{2} + \frac{\sigma_1 + \sigma_2}{2} \cos 2\theta \qquad \left(\because OE = \text{Radius} = \frac{\sigma_1 + \sigma_2}{2}\right)$$

$= \sigma_n$ or Normal stress

and
$$ED = OE \sin 2\theta$$

$$= \frac{\sigma_1 + \sigma_2}{2} \sin 2\theta \qquad \left(\because OE = \frac{\sigma_1 + \sigma_2}{2}\right)$$

$= \sigma_t$ or Tangential (or shear) stress.

3.5.3. Mohr's Circle when a Body is Subjected to two Mutually Perpendicular Principal Tensile Stresses Accompanied by a Simple Shear Stress. Consider a rectangular body subjected to two mutually perpendicular principal tensile stresses of unequal intensities accompanied by a simple shear stress. It is required to find the resultant stress on an oblique plane as shown in Fig. 3.22.

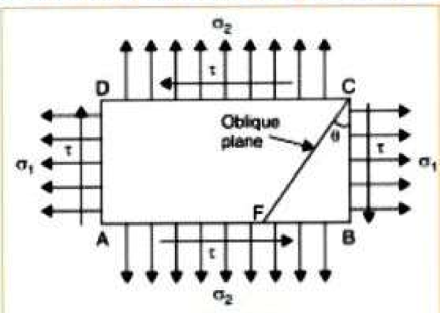

Fig. 3.22

Let σ_1 = Major tensile stress

σ_2 = Minor tensile stress

τ = Shear stress across face *BC* and *AD*

θ = Angle made by the oblique plane with the plane of major tensile stress.

According to the principle of shear stress, the faces *AB* and *CD* will also be subjected to a shear stress of τ.

Mohr's circle is drawn as given in Fig. 3.23.

Take any point *A* and draw a horizontal line through *A*.

Take $AB = \sigma_1$ and $AC = \sigma_2$ towards right of *A* to some suitable scale. Draw perpendiculars at *B* and *C* and cut off *BF* and *CG* equal to shear stress τ to the same scale. Bisect *BC* at *O*. Now with *O* as centre and radius equal to *OG* or *OF* draw a circle. Through *O*, draw a line *OE* making an angle of 2θ with *OF* as shown in Fig. 3.23. From *E*, draw *ED* perpendicular to *CB*. Join *AE*. Then length *AE* represents the resultant stress on the given oblique plane. And lengths *AD* and *ED* represents the normal stress and tangential stress respectively.

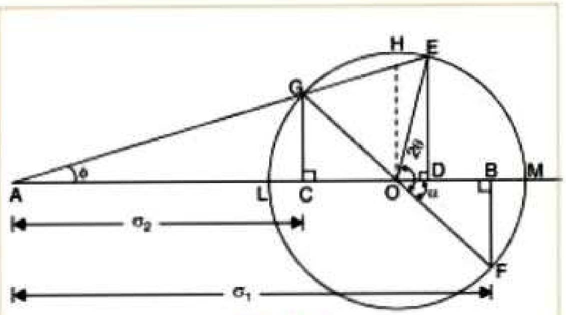

Fig. 3.23

Hence from Fig. 3.23, we have

Length *AE* = Resultant stress on the oblique plane

Length *AD* = Normal stress on the oblique plane

Length *ED* = Shear stress on the oblique plane.

Proof. (See Fig. 3.23).

$$CO = \frac{1}{2} CB = \frac{1}{2} [\sigma_1 - \sigma_2] \qquad (\because \; CB = \sigma_1 - \sigma_2)$$

$$AO = AC + CO = \sigma_2 + \frac{1}{2} [\sigma_1 - \sigma_2]$$

$$= \frac{2\sigma_2 + \sigma_1 - \sigma_2}{2} = \frac{\sigma_1 + \sigma_2}{2}$$

$$AD = AO + OD$$

$$= \frac{\sigma_1 + \sigma_2}{2} + OE \cos (2\theta - \alpha) \qquad [\because \; OD = OE \cos (2\theta - \alpha)]$$

$$= \frac{\sigma_1 + \sigma_2}{2} + OE [\cos 2\theta \cos \alpha + \sin 2\theta \sin \alpha]$$

$$= \frac{\sigma_1 + \sigma_2}{2} + OE \cos 2\theta \cos \alpha + OE \sin 2\theta \sin \alpha$$

$$= \frac{\sigma_1 + \sigma_2}{2} + OE \cos \alpha \,.\, \cos 2\theta + OE \sin \alpha \,.\, \sin 2\theta$$

$$= \frac{\sigma_1 + \sigma_2}{2} + OF \cos \alpha \,.\, \cos 2\theta + OF \sin \alpha \,.\, \sin 2\theta$$

$$(\because \; OE = OF = \text{Radius})$$

$$= \frac{\sigma_1 + \sigma_2}{2} + OB \cos 2\theta + BF \sin 2\theta \qquad (\because \ OF \cos \alpha = OB, \ OF \sin \alpha = BF)$$

$$= \frac{\sigma_1 + \sigma_2}{2} + CO \cos 2\theta + \tau \sin 2\theta \qquad (\because \ OB = CO, \ BF = \tau)$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta + \tau \sin 2\theta \qquad \left(\because \ CO = \frac{\sigma_1 - \sigma_2}{2}\right)$$

$= \sigma_n$ or Normal stress

Now $ED = OE \sin (2\theta - \alpha) = OE (\sin 2\theta \cos \alpha - \cos 2\theta \sin \alpha)$

$= OE \sin 2\theta \cos \alpha - OE \cos 2\theta \sin \alpha$

$= OE \cos \alpha \,.\, \sin 2\theta - OE \sin \alpha \,.\, \cos 2\theta$

$= OE \cos \alpha \,.\, \sin 2\theta - OE \sin \alpha \,.\, \cos 2\theta \qquad (\because \ OE = OF = \text{Radius})$

$= OB \,.\, \sin 2\theta - BF \cos 2\theta \qquad (\because \ OF \cos \alpha = OB, \ OF \sin \alpha = BF)$

$= CO \,.\, \sin 2\theta - \tau \cos 2\theta \qquad (\because \ OB = CO, \ BF = \tau)$

$$= \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta - \tau \cos 2\theta \qquad \left(\because \ CO = \frac{\sigma_1 - \sigma_2}{2}\right)$$

$= \sigma_t$ or Tangential stress.

Maximum and minimum value of normal stress. In Fig. 3.23, the normal stress is given by *AD*. Hence the maximum value of *AD* will be when *D* coincides with *M* and minimum value of *AD* will be when *D* coincides with *L*.

∴ Maximum value of normal stress,

$$(\sigma_n)_{max} = AM = AO + OM$$

$$= \frac{\sigma_1 + \sigma_2}{2} + OF \qquad \left(\because \ AO = \frac{\sigma_1 + \sigma_2}{2}, OM = OF = \text{Radius}\right)$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \sqrt{OB^2 + BF^2} \qquad (\because \ \text{In triangle } OBF, \ OF = \sqrt{OB^2 + BF^2})$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2} \qquad \left(\because \ OB = \frac{\sigma_1 - \sigma_2}{2}, BF = \tau\right)$$

Minimum value of normal stress,

$$(\sigma_n)_{min} = AL = AO - LO$$

$$= \frac{\sigma_1 + \sigma_2}{2} - OF \qquad (\because \ LO = OF = \text{Radius})$$

$$= \frac{\sigma_1 + \sigma_2}{2} - \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2}$$

(*i*) For maximum normal stress, the point *D* coincides with *M*. But when the point *D* coincides with *M*, the point *E* also coincides with *M*. Hence for maximum value of normal stress,

Angle $2\theta = \alpha \qquad (\because$ Line *OE* coincides with line *OM*)

$$\therefore \quad \theta = \frac{\alpha}{2} \qquad ...(i)$$

Also
$$\tan 2\theta = \tan \alpha = \frac{BF}{OB} = \frac{\tau}{\frac{\sigma_1 - \sigma_2}{2}} \qquad \left(\because \ BF = \tau, OB = \frac{\sigma_1 - \sigma_2}{2}\right)$$

$$= \frac{2\tau}{\sigma_1 - \sigma_2}.$$

(*ii*) For maximum and minimum normal stresses, the shear stress is zero and hence the planes, on which maximum and minimum normal stresses act, are known as *principal planes* and the stresses are known as *principal stresses*.

(*iii*) For minimum normal stress, the point *D* coincides with point *L*. But when the point *D* coincides with *L*, the point *E* also coincides with *L*. Then

Angle $2\theta = \pi + \alpha$ (∵ Line *OE* coincides with line *OL*)

∴ $$\theta = \frac{\pi}{2} + \frac{\alpha}{2} \qquad ...(ii)$$

From equations (*i*) and (*ii*), it is clear that the plane of minimum normal stress is inclined at an angle 90° to the plane of maximum normal stress.

Maximum value of shear stress. Shear stress is given by *ED*. Hence maximum value of *ED* will be when *E* coincides with *G*, and *D* coincides with *O*.

∴ Maximum shear stress,

$$(\sigma_t)_{max} = OH = OF \qquad (\because \; OH = OF = \text{radius})$$

$$= \sqrt{OB^2 + BF^2} \qquad (\because \; \text{In triangle } OBF,\ OF = \sqrt{OB^2 + BF^2})$$

$$= \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2} \qquad \left(\because \; OB = \frac{\sigma_1 - \sigma_2}{2},\ BF = \tau\right)$$

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LECTURER NOTES
3RD SEMESTER
MECHANICAL ENGINEERING

2022

Strength of Material

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Chapter- 04

BENDING MOMENT AND SHEAR FORCE

Beam: - Beam is a structural member which is subjected to transverse loading.

or

A structural member which is acted upon by a system of external loads at right angles to its axis is known as beam.

Types of Beam: - The beams are classified into the following types

1. Cantilever beam
2. Simply supported beam
3. Overhang beam
4. Fixed beam or built-in beam
5. Continuous beam

Cantilever beam: - A beam which is fixed at one end and free at other end, is known as cantilever beam, shown in the Fig.6.1

Fig. 6.1

Simply supported beam: - A beam supported or resting freely on the supports at its both ends, is known as simply supported beam, shown in the Fig.6.2

Fig. 6.2

Overhang beam: - If the end portion of the beam is extended beyond the support, such beam is known as Overhang beam, shown in the Fig.6.3

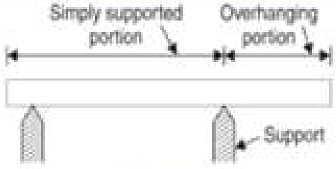


Fig. 6.3

Fixed beam or built-in beam: - A beam whose both ends are fixed or built-in walls, is known as Fixed beam. A Fixed beam is also known as built-in or encastred beam,shown in the Fig 6.4

Fig. 6.4

. **Continuous beam: -** A beam which is provided with more than two supports is known as continuous beam, shown in the Fig 6.5

Fig. 6.5

Types of Load:-

A beam is normally horizontal and the loads acting on the beams are generally vertical. The following are the important types of load acting on a beam :

1. Concentrated or point load,
2. Uniformly distributed load, and
3. Uniformly varying load.

6.4.1. Concentrated or Point Load. A concentrated load is one which is considered to act at a point, although in practice it must really be distributed over a small area. In Fig. 6.6, W shows the point load.

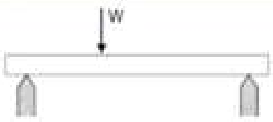


Fig. 6.6

Fig. 6.7

6.4.2. Uniformly Distributed Load. A uniformly distributed load is one which is spread over a beam in such a manner that rate of loading w is uniform along the length (*i.e.*, each unit length is loaded to the same rate) as shown in Fig. 6.7. The rate of loading is expressed as w N/m run. Uniformly distributed load is, represented by u.d.l.

For solving the numerical problems, the total uniformly distributed load is converted into a point load, acting at the centre of uniformly distributed load.

6.4.3. Uniformly Varying Load. A uniformly varying load is one which is spread over a beam in such a manner that rate of loading varies from point to point along the beam as shown in Fig. 6.8 in which load is zero at one end and increases uniformly to the other end. Such load is known as triangular load.

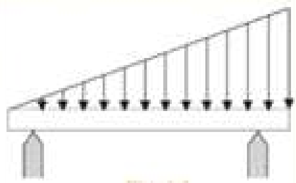


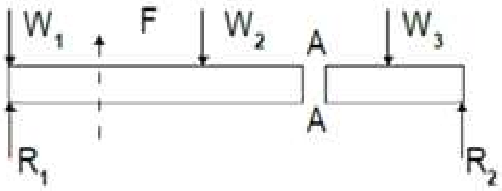
Fig. 6.8

For solving numerical problems the total load is equal to the area of the triangle and this total load is assumed to be acting at the C.G. of the triangle *i.e.*, at a distance of $\frac{2}{3}$rd of total length of beam from left end.

Concepts of shear force and bending moment

Shear force

The shearing force at any section of beam represents the tendency for the portion of beam to one side of the section of slide or shear laterally relative to the other portion.

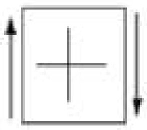


The resultant of the loads and reactions to the left of A is vertically upwards and the since the whole became is in equilibrium, the resultant of the forces to the right of AA must also be F acting down ward. F is called the shearing force.

Definition

The algebraic sum of all the vertical forces either left or right of the section of the beam is known as Shear Force. It is denoted by S.F.

Shearing force will be considered positive when the resultant of the forces to the left is upwards or to the right in downward.



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Concepts of Bending Moment

Bending Moment

The algebraic sum of moments of all the vertical forces acting either left or right of the section of the beam is known as Bending Moment.it is denoted by B.M.

In a small manner it can be argued that if the moment about the section AA of the forces to the left is M clockwise then the moment of the forces to the right of AA must be anticlockwise. M is called the bending moment.

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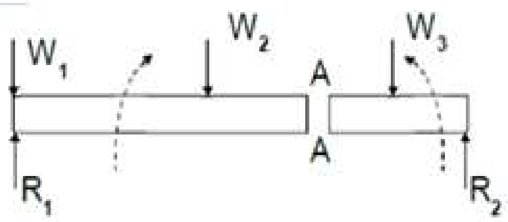

Definition

The algebraic sum of moments of all the vertical forces acting either left or right of the section of the beam is known as Bending Moment.it is denoted by B.M.

Bending moment will be considered positive when the moment on the left of section is Clockwise and on the right portion anticlockwise. This is referred as sagging the beam because concave upwards. Negative B.M is termed as hogging. A BMD is one which shows the variation of bending moment along the length of the beam.

Shear Force Diagram and Bending Moment Diagram

A shear force diagram is one which shows the variation of the shear force along the length of the beam. And a bending moment diagram is one which shows the variation of the bending moment along the length of the beam.

IMPORTANT POINTS FOR DRAWING SHEAR FORCE AND BENDING MOMENT DIAGRAMS

In Art. 6.2, it is mentioned that the shear force diagram is one which shows the variation of the shear force along the length of the beam. And a bending moment diagram is one which show the variation of the bending moment along the length of beam. In these diagrams, the shear force or bending moment are represented by ordinates whereas the length of the beam represents abscissa.

The following are the important points for drawing shear force and bending moment diagrams :

1. Consider the left or the right portion of the section.

2. Add the forces (including reaction) normal to the beam on one of the portion. If right portion of the section is chosen, a force on the right portion acting downwards is positive while a force acting upwards is negative.

If the left portion of the section is chosen, a force on the left portion acting upwards is positive while a force acting downwards is negative.

3. The positive values of shear force and bending moments are plotted above the base line, and negative values below the base line.

4. The shear force diagram will increase or decrease suddenly *i.e.*, by a vertical straight line at a section where there is a vertical point load.

5. The shear force between any two vertical loads will be constant and hence the shear force diagram between two vertical loads will be horizontal.

6. The bending moment at the two supports of a simply supported beam and at the free end of a cantilever will be zero.

SHEAR FORCE AND BENDING MOMENT DIAGRAMS FOR A CANTILEVER WITH A POINT LOAD AT THE FREE END

Fig. 6.14 shows a cantilever *AB* of length *L* fixed at *A* and free at *B* and carrying a point load *W* at the free end *B*.

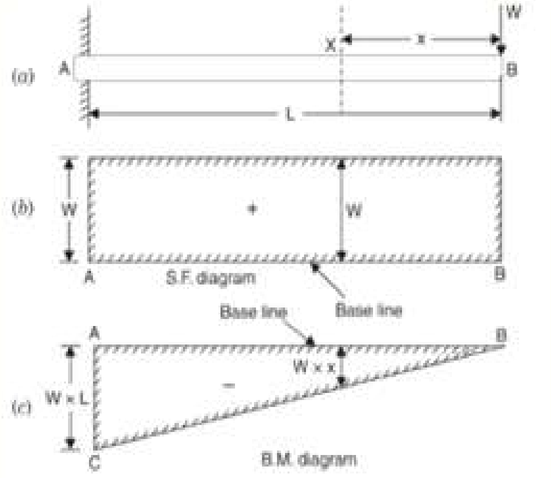

Fig. 6.14

Let F_x = Shear force at *X*, and

M_x = Bending moment at *X*.

Take a section *X* at a distance *x* from the free end. Consider the right portion of the section.

The shear force at this section is equal to the resultant force acting on the right portion at the given section. But the resultant force acting on the right portion at the section *X* is *W* and acting in the downward direction. But a force on the right portion acting downwards is considered positive. Hence shear force at *X* is positive.

$\therefore \quad F_x = +W$

The shear force will be constant at all sections of the cantilever between *A* and *B* as there is no other load between *A* and *B*. The shear force diagram is shown in Fig. 6.14 (*b*).

Bending Moment Diagram

The bending moment at the section *X* is given by

$$M_x = -W \times x \qquad ...(i)$$

(Bending moment will be negative as for the right portion of the section, the moment of *W* at *X* is clockwise. Also the bending of cantilever will take place in such a manner that convexity will be at the top of the beam).

From equation (*i*), it is clear that B.M. at any section is proportional to the distance of the section from the free end.

At $x = 0$ *i.e.*, at *B*, B.M. = 0

At $x = L$ *i.e.*, at *A*, B.M. = $W \times L$

Hence B.M. follows the straight line law. The B.M. diagram is shown in Fig. 6.14 (*c*). At point *A*, take $AC = W \times L$ in the downward direction. Join point *B* to *C*.

The shear force and bending moment diagrams for several concentrated loads acting on a cantilever, will be drawn in the similar manner.

SHEAR FORCE AND BENDING MOMENT DIAGRAMS FOR A CANTILEVER WITH A UNIFORMLY DISTRIBUTED LOAD

Fig. 6.16 shows a cantilever of length L fixed at A and carrying a uniformly distributed load of w per unit length over the entire length of the cantilever.

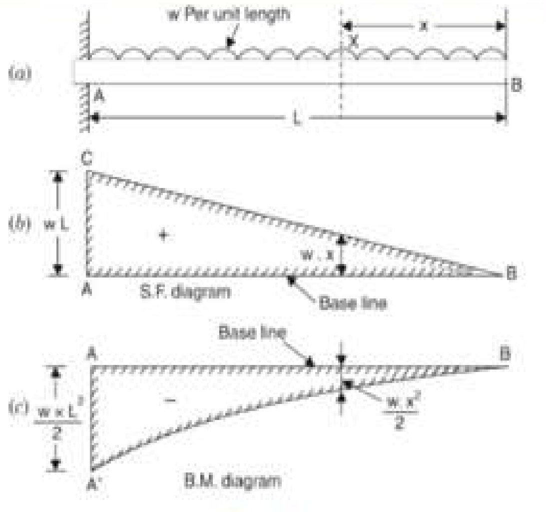

Fig. 6.16

Take a section X at a distance of x from the free end B.

Let F_x = Shear force at X, and

M_x = Bending moment at X.

Here we have considered the right portion of the section. The shear force at the section X will be equal to the resultant force acting on the right portion of the section. But the resultant force on the right portion = w × Length of right portion = $w.x$.

This resultant force is acting downwards. But the resultant force on the right portion acting downwards is considered positive. Hence shear force at X is positive.

$\therefore \quad F_x = + w.x$

The above equation shows that the shear force follows a straight line law.

At B, $x = 0$ and hence $\quad F_x = 0$

At A, $x = L$ and hence $\quad F_x = w.L$

The shear force diagram is shown in Fig. 6.16 (*b*).

Bending Moment Diagram

It is mentioned in Art. 6.4.3 that the uniformly distributed load over a section is converted into point load acting at the C.G. of the section.

The bending moment at the section X is given by

M_x = − (Total load on right portion) × Distance of C.G. of right portion from X

$$= -(w \,.\, x) \,.\, \frac{x}{2} = -w \,.\, x \,.\, \frac{x}{2} = -w \,.\, \frac{x^2}{2} \qquad ...(i)$$

(The bending moment will be negative as for the right portion of the section, the moment of the load at x is clockwise. Also the bending of cantilever will take place in such a manner that convexity will be at the top of the cantilever).

From equation (*i*), it is clear that B.M. at any section is proportional to the square of the distance of the section from the free end. This follows a parabolic law.

At B, $x = 0$ hence $\quad M_x = 0$

At A, $x = L$ hence $\quad M_x = -w \,.\, \dfrac{L^2}{2}$.

Problem 6.1. *A cantilever beam of length 2 m carries the point loads as shown in Fig. 6.15. Draw the shear force and B.M. diagrams for the cantilever beam.*

Sol. Given :

Refer to Fig. 6.15.

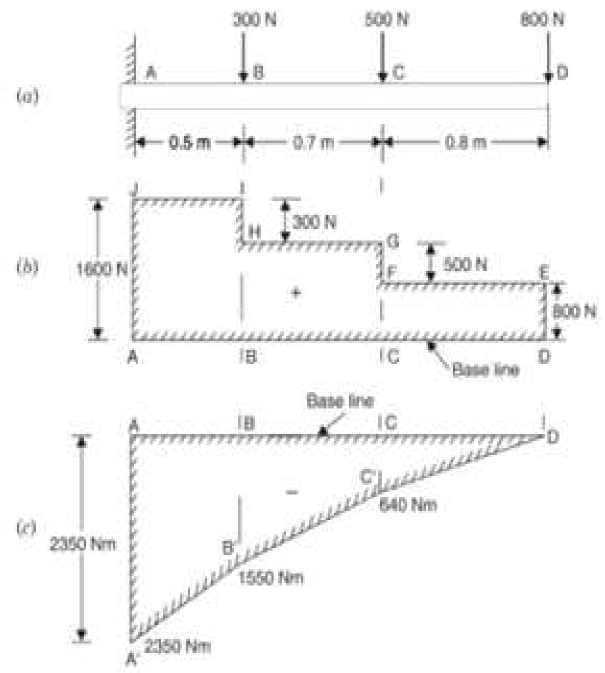

Fig. 6.15

Shear Force Diagram

The shear force at D is + 800 N. This shear force remains constant between D and C. At C, due to point load, the shear force becomes (800 + 500) = 1300 N. Between C and B, the shear force remains 1300 N. At B again, the shear force becomes (1300 + 300) = 1600 N. The shear force between B and A remains constant and equal to 1600 N. Hence the shear force at different points will be as follows :

S.F. at D, $F_D = +800$ N

S.F. at C, $F_C = +800 + 500 = +1300$ N

S.F. at B, $F_B = +800 + 500 + 300 = 1600$ N

S.F. at A, $F_A = +1600$ N.

The shear force, diagram is shown in Fig. 6.15 (*b*) which is drawn as :

Draw a horizontal line AD as base line. On the base line mark the points B and C below the point loads. Take the ordinate DE = 800 N in the upward direction. Draw a line EF parallel to AD. The point F is vertically above C. Take vertical line FG = 500 N. Through G, draw a horizontal line GH in which point H is vertically above B. Draw vertical line HI = 300 N. From I, draw a horizontal line IJ. The point J is vertically above A. This completes the shear force diagram.

Bending Moment Diagram

The bending moment at D is zero :

(*i*) The bending moment at any section between C and D at a distance x and D is given by,

$$M_x = -800 \times x \text{ which follows a straight line law.}$$

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At C, the value of $x = 0.8$ m.

$\therefore$ B.M. at C, $M_C = -800 \times 0.8 = -640$ Nm.

(*ii*) The B.M. at any section between B and C at a distance x from D is given by (At C, $x = 0.8$ and at B, $x = 0.8 + 0.7 = 1.5$ m. Hence here x varies from 0.8 to 1.5).

$$M_x = -800\,x - 500\,(x - 0.8) \quad \text{...(i)}$$

Bending moment between B and C also varies by a straight line law.

B.M. at B is obtained by substituting $x = 1.5$ m in equation (*i*),

$$\therefore \quad M_B = -800 \times 1.5 - 500\,(1.5 - 0.8)$$
$$= -1200 - 350 = -1550 \text{ Nm.}$$

(*iii*) The B.M. at any section between A and B at a distance x from D is given by (At B, $x = 1.5$ and at A, $x = 2.0$ m. Hence here x varies from 1.5 m to 2.0 m)

$$M_x = -800\,x - 500\,(x - 0.8) - 300\,(x - 1.5) \quad \text{...(ii)}$$

Bending moment between A and B varies by a straight line law.

B.M. at A is obtained by substituting $x = 2.0$ m in equation (*ii*),

$$\therefore \quad M_A = -800 \times 2 - 500\,(2 - 0.8) - 300\,(2 - 1.5)$$
$$= -800 \times 2 - 500 \times 1.2 - 300 \times 0.5$$
$$= -1600 - 600 - 150 = -2350 \text{ Nm.}$$

Hence the bending moments at different points will be as given below :

$$M_D = 0$$
$$M_C = -640 \text{ Nm}$$
$$M_B = -1550 \text{ Nm}$$
and $$M_A = -2350 \text{ Nm.}$$

The bending moment diagram is shown in Fig. 6.15 (*c*) which is drawn as.

Draw a horizontal line AD as a base line and mark the points B and C on this line. Take vertical lines $CC' = 640$ Nm, $BB' = 1550$ Nm and $AA' = 2350$ Nm in the downward direction. Join points D, C', B' and A' by straight lines. This completes the bending moment diagram.

Problem 6.2. *A cantilever of length 2.0 m carries a uniformly distributed load of 1 kN/m run over a length of 1.5 m from the free end. Draw the shear force and bending moment diagrams for the cantilever.*

Sol. Given :

U.D.L., $w = 1$ kN/m run

Refer to Fig. 6.17.

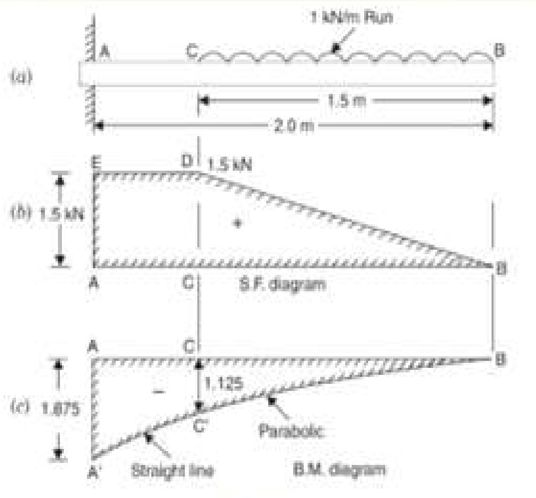

Fig. 6.17

Shear Force Diagram

Consider any section between C and B a distance of x from the free end B. The shear force at the section is given by

$F_x = w.x$ (+ve sign is due to downward force on right portion of the section)

$= 1.0 \times x$ ($\because$ $w = 1.0$ kN/m run)

At B, $x = 0$ hence $F_x = 0$

At C, $x = 1.5$ hence $F_x = 1.0 \times 1.5 = 1.5$ kN.

The shear force follows a straight line law between C and B. As between A and C there is no load, the shear force will remain constant. Hence shear force between A and C will be represented by a horizontal line.

The shear force diagram is shown in Fig. 6.17 (*b*) in which

$$F_B = 0, F_C = 1.5 \text{ kN and } F_A = F_C = 1.5 \text{ kN.}$$

Bending Moment Diagram

(*i*) The bending moment at any section between C and B at a distance x from the free end B is given by

$$M_x = -(w.x.).\frac{x}{2} = -\left(1.\frac{x^2}{2}\right) = -\frac{x^2}{2} \quad ...(i)$$

(The bending moment will be negative as for the right portion of the section the moment of load at x is clockwise).

At B, $x = 0$ hence $M_B = -\frac{0^2}{2} = 0$

At C, $x = 1.5$ hence $M_C = -\frac{1.5^2}{2} = -1.125$ Nm

From equation (*i*) it is clear that the bending moment varies according to parabolic law between C and B.

(*ii*) The bending moment at any section between A and C at a distance x from the free end B is obtained as : (here x varies from 1.5 m to 2.0 m)

Total load due to U.D.L.= $w \times 1.5 = 1.5$ kN.

This load is acting at a distance of $\frac{1.5}{2} = 0.75$ m from the free end B or at a distance of $(x - 0.75)$ from any section between A and C.

$\therefore$ Moment of this load at any section between A and C at a distance x from free end

= (Load due to U.D.L.) $\times (x - 0.75)$

$\therefore$ $M_x = -1.5 \times (x - 0.75)$ ...(*ii*)

(– ve sign is due to clockwise moment for right portion)

From equation (*ii*) it is clear that the bending moment follows straight line law between A and C.

At C, $x = 1.5$ m hence $M_C = -1.5\,(1.5 - 0.75) = -1.125$ Nm

At A, $x = 2.0$ m hence $M_A = -1.5\,(2 - 0.75) = -1.875$ Nm.

Now the bending moment diagram is drawn as shown in Fig. 6.17 (*c*). In this diagram line CC' = 1.125 Nm and AA' = 1.875 Nm. The points B and C' are on a parabolic curve whereas the points A' and C' are joined by a straight line.

SHEAR FORCE AND BENDING MOMENT DIAGRAMS FOR A CANTILEVER CARRYING A GRADUALLY VARYING LOAD

Fig. 6.22 shows a cantilever of length L fixed at A and carrying a gradually varying load from zero at the free end to w per unit length at the fixed end.

Fig. 6.22 shows a cantilever of length L fixed at A and carrying a gradually varying load from zero at the free end to w per unit length at the fixed end.

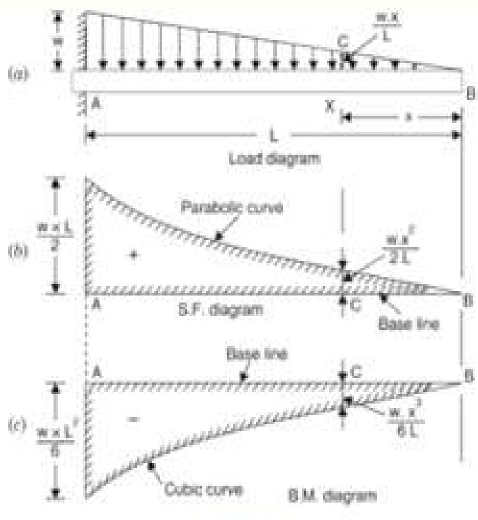

Fig. 6.22

Take a section X at a distance x from the free end B.

Let F_x = Shear force at the section X, and

M_x = Bending moment at the section X.

Let us first find the rate of loading at the section X. The rate of loading is zero at B and is w per metre run at A. This means that rate of loading for a length L is w per unit length. Hence rate of loading for a length of x will be $\frac{w}{L} \times x$ per unit length. This is shown in Fig. 6.22 (a) by CX, which is also known as load diagram. Hence $CX = \frac{w.x}{L}$.

The shear force and the section X at a distance x from free end is given by,

$$F_x = \text{Total load on the cantilever for a length } x \text{ from the free end } B$$

$$= \text{Area of triangle } BCX$$

$$= \frac{XB \cdot XC}{2} = \frac{x\left(\frac{w.x}{L}\right)}{2} \qquad \left(\because\ XB = x, XC = \frac{w.x}{L}\right)$$

$$= \frac{w.x^2}{2L} \qquad \ldots(i)$$

Equation (i) shows that the S.F. varies according to the parabolic law.

At B, $x = 0$ hence $F_B = \frac{w \times 0^2}{2L} = 0$

At A, $x = L$ hence $F_A = \frac{w.L^2}{2L} = \frac{w.L}{2}$

The bending moment at the section X at a distance x from the free end B is given by,

$$M_x = -(\text{Total load for a length } x) \times \text{Distance of the load from } X$$

$$= -(\text{Area of triangle } BCX) \times \text{Distance of C.G. of the triangle from } X$$

$$= -\left(\frac{wx^2}{2L}\right) \times \frac{x}{3} = -\frac{wx^3}{6L} \qquad \ldots(ii)$$

Equation (*ii*) shows that the B.M. varies according to the cubic law.

At B, $x = 0$ hence $\quad M_B = -\frac{w \times 0}{6L} = 0$

At A, $x = L$ hence $\quad M_A = -\frac{w.L^3}{6L} = -\frac{w.L^2}{6}.$

Problem 6.7. *A cantilever of length 4 m carries a gradually varying load, zero at the free end to 2 kN/m at the fixed end. Draw the S.F. and B.M. diagrams for the cantilever.*

Sol. Given :

Length, $\quad L = 4$ m

Load at fixed end, $\quad w = 2$ kN/m

Shear Force Diagram

The shear force is zero at B. The shear force at C will be equal to the area of load diagram ABC.

$\therefore \quad$ Shear force at $C = \frac{4 \times 2}{2} = 4$ kN

The shear force between A and B varies according to parabolic law.

Bending Moment Diagram

The B.M. at B is zero. The bending moment at A is equal to $-\frac{w.L^2}{6}$.

$$\therefore \quad M_A = -\frac{w.L^2}{6} = -\frac{2 \times 4^2}{6} = -5.33 \text{ kNm}.$$

The B.M. between A and B varies according to cubic law.

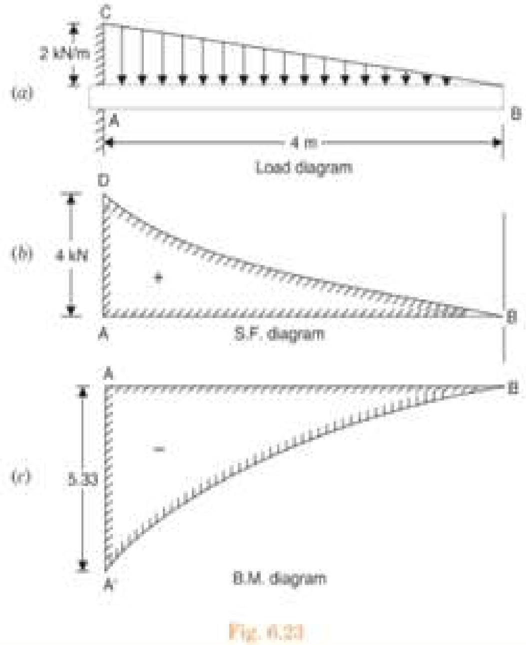

Fig. 6.23

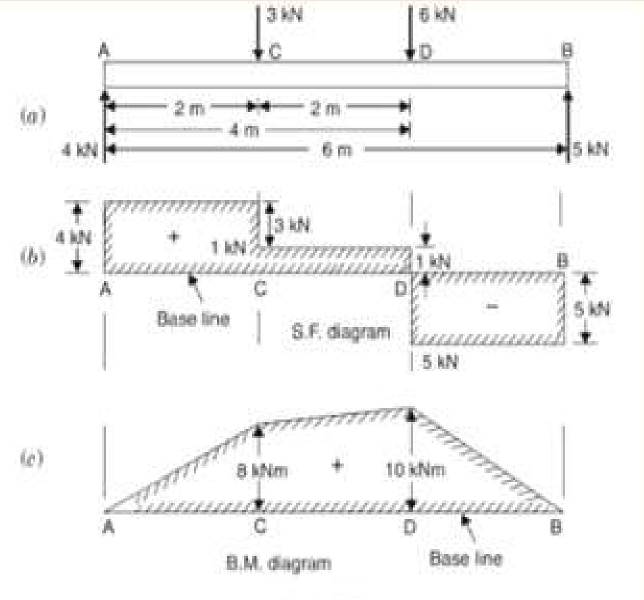

Fig. 6.26

Bending Moment Diagram

B.M. at A, $M_A = 0$

B.M. at C, $M_C = R_A \times 2 = 4 \times 2 = +8$ kNm

B.M. at D, $M_D = R_A \times 4 - 3 \times 2 = 4 \times 4 - 3 \times 2 = +10$ kNm

B.M. at B, $M_B = 0$

The bending moment diagram is drawn as shown in Fig. 6.26 (c).

Problem 6.9. *Draw the shear force and bending moment diagram for a simply supported beam of length 9 m and carrying a uniformly distributed load of 10 kN/m for a distance of 6 m from the left end. Also calculate the maximum B.M. on the section.*

Sol. First calculate reactions R_A and R_B.

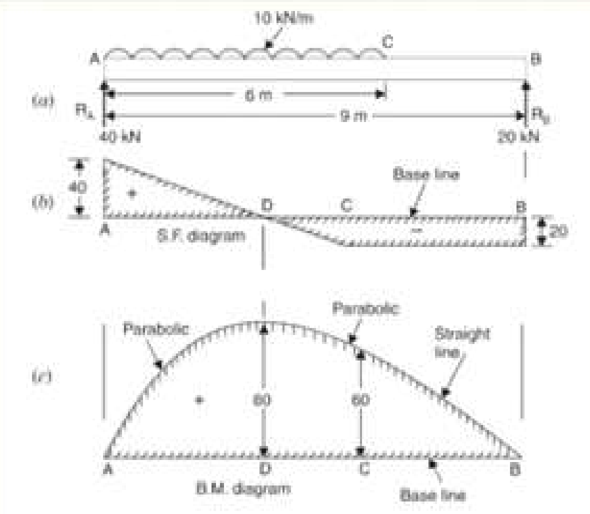

Fig. 6.28

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Taking moments of the forces about A, we get

$$R_B \times 9 = 10 \times 6 \times \frac{6}{2} = 180$$

$$\therefore \quad R_B = \frac{180}{9} = 20 \text{ kN}$$

$$\therefore \quad R_A = \text{Total load on beam} - R_B = 10 \times 6 - 20 = 40 \text{ kN.}$$

Shear Force Diagram

Consider any section at a distance x from A between A and C. The shear force at the section is given by,

$$F_x = +R_A - 10\,x = +40 - 10\,x \quad \text{...(i)}$$

Equation (*i*) shows that shear force varies by a straight line law between A and C.

At A, $x = 0$ hence $\quad F_A = +40 - 0 = 40$ kN

At C, $x = 6$ m hence $\quad F_C = +40 - 10 \times 6 = -20$ kN

The shear force at A is $+40$ kN and at C is -20 kN. Also shear force between A and C varies by a straight line. This means that somewhere between A and C, the shear force is zero. Let the S.F. is zero at x metre from A. Then substituting the value of S.F. (*i.e.*, F_x) equal to zero in equation (*i*), we get

$$0 = 40 - 10x$$

$$\therefore \quad x = \frac{40}{10} = 4 \text{ m}$$

Hence shear force is zero at a distance 4 m from A.

The shear force is constant between C and B. This equal to -20 kN.

Now the shear force diagram is drawn as shown in Fig. 6.28 (*b*). In the shear force diagram, distance $AD = 4$ m. The point D is at a distance 4 m from A.

B.M. Diagram

The B.M. at any section between A and C at a distance x from A is given by,

$$M_x = R_A \times x - 10 \,.\, x \,.\, \frac{x}{2} = 40x - 5x^2 \quad \text{...(ii)}$$

Equation (*ii*) shows that B.M. varies according to parabolic law between A and C.

At A, $x = 0$ hence $\quad M_A = 40 \times 0 - 5 \times 0 = 0$

At C, $x = 6$ m hence $\quad M_C = 40 \times 6 - 5 \times 6^2 = 240 - 180 = +60$ kNm

At D, $x = 4$ m hence $\quad M_D = 40 \times 4 - 5 \times 4^2 = 160 - 80 = +80$ kNm

The bending moment between C and B varies according to linear law.

B.M. at B is zero whereas at C is 60 kNm.

The bending moment diagram is drawn as shown in Fig. 6.28 (*c*).

Maximum Bending Moment

The B.M. is maximum at a point where shear force changes sign. This means that the point where shear force becomes zero from positive value to the negative or *vice-versa*, the B.M. at that point will be maximum. From the shear force diagram, we know that at point D, the shear force is zero after changing its sign. Hence B.M. is maximum at point D. But the B.M. at D is $+80$ kNm.

$\therefore$ Max. B.M. = **+ 80 kN. Ans.**

Problem 6.10. *Draw the shear force and B.M. diagrams for a simply supported beam of length 8 m and carrying a uniformly distributed load of 10 kN/m for a distance of 4 m as shown in Fig. 6.29.*

Sol. First calculate the reactions R_A and R_B.

Taking moments of the forces about A, we get

$$R_B \times 8 = 10 \times 4 \times \left(1 + \frac{4}{2}\right) = 120$$

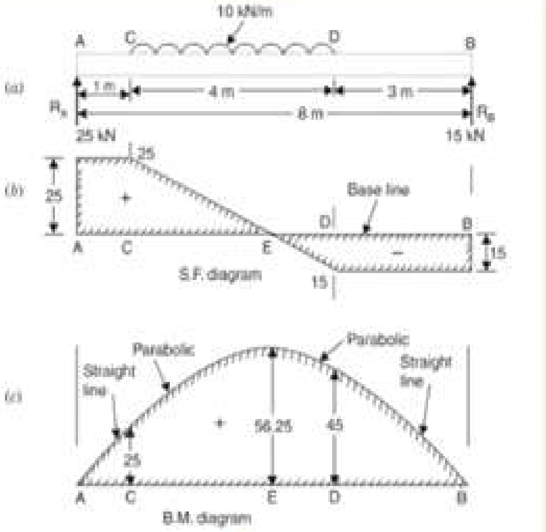

Fig. 6.29

$$\therefore \quad R_B = \frac{120}{8} = 15 \text{ kN}$$

$$\therefore \quad R_A = \text{Total load on beam} - R_B$$
$$= 10 \times 4 - 15 = 25 \text{ kN}$$

Shear Force Diagram

The shear force at A is + 25 kN. The shear force remains constant between A and C and equal to + 25 kN. The shear force at B is – 15 kN. The shear force remains constant between B and D and equal to – 15 kN. The shear force at any section between C and D at a distance x from A is given by,

$$F_x = +25 - 10(x-1) \quad ...(i)$$

At C, $x = 1$ hence $\quad F_C = +25 - 10(1-1) = +25$ kN

At D, $x = 5$ hence $\quad F_D = +25 - 10(5-1) = -15$ kN

The shear force at C is + 25 kN and at D is – 15 kN. Also shear force between C and D varies by a straight line law. This means that somewhere between C and D, the shear force is zero. Let the S.F. be zero at x metre from A. Then substituting the value of S.F. (*i.e.*, F_x) equal to zero in equation (*i*), we get

$$0 = 25 - 10(x-1)$$

or $\quad 0 = 25 - 10x + 10 \quad$ or $\quad 10x = 35$

$$\therefore \quad x = \frac{35}{10} = 3.5 \text{ m}$$

Hence the shear force is zero at a distance 3.5 m from A.

Hence the distance AE = 3.5 m in the shear force diagram shown in Fig. 6.29 (*b*).

B.M. Diagram

B.M. at A is zero

B.M. at B is also zero

B.M. at $C = R_A \times 1 = 25 \times 1 = 25$ kNm

The B.M. at any section between C and D at a distance x from A is given by,

$$M_x = R_A \cdot x - 10(x-1) \cdot \frac{(x-1)}{2} = 25 \times x - 5(x-1)^2 \quad ...(ii)$$

At C, $x = 1$ hence $M_C = 25 \times 1 - 5(1-1)^2 = 25$ kNm

At D, $x = 5$ hence $M_D = 25 \times 5 - 5(5-1)^2 = 125 - 80 = 45$ kNm

At E, $x = 3.5$ hence $M_E = 25 \times 3.5 - 5(3.5-1)^2 = 87.5 - 31.25 = 56.25$ kNm

B.M. will increase from 0 at A to 25 kNm at C by a straight line law. Between C and D the B.M. varies according to parabolic law as is clear from equation (*ii*). Between C and D, the B.M. will be maximum at E. From D to B the B.M. will decrease from 45 kNm at D to zero at B according to straight line law.

SHEAR FORCE AND BENDING MOMENT DIAGRAMS FOR OVER-HANGING BEAMS

If the end portion of a beam is extended beyond the support, such beam is known as overhanging beam. In case of overhanging beams, the B.M. is positive between the two supports, whereas the B.M. is negative for the over-hanging portion. Hence at some point, the B.M. is zero after changing its sign from positive to negative or *vice versa*. That point is known as the *point of contraflexure* or *point of inflexion*.

6.15.1. Point of Contraflexure. It is the point where the B.M. is zero after changing its sign from positive to negative or *vice versa*.

Problem 6.14. *Draw the shear force and bending moment diagrams for the over-hanging beam carrying uniformly distributed load of 2 kN/m over the entire length as shown in Fig. 6.35. Also locate the point of contraflexure.*

Sol. First calculate the reactions R_A and R_B

Taking moments of all forces about A, we get

$$R_B \times 4 = 2 \times 6 \times \frac{6}{2} = 36$$

(∵ Total load on beam = 2 × 6 = 12 kN. This load is acting at a distance 3 m from A)

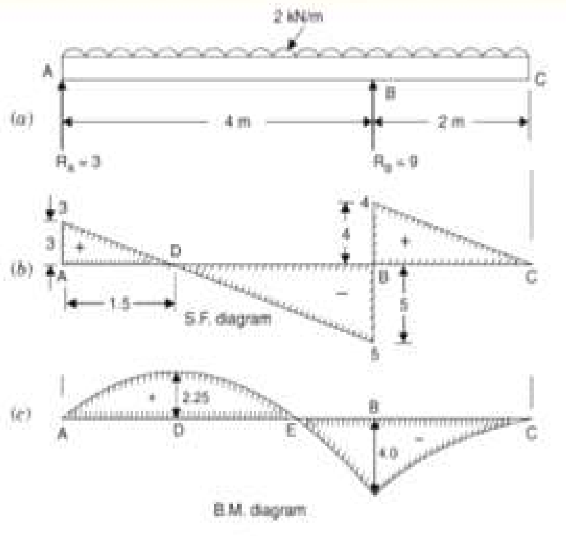

Fig. 6.35

$$\therefore \quad R_B = \frac{36}{4} = 9 \text{ kN}$$

and $\quad R_A = \text{Total load} - R_B = 2 \times 6 - 9 = 3$ kN

Pi

Shear Force Diagram

Shear force at $A = +R_A = +3$ kN

(*i*) The shear force at any section between A and B at a distance x from A is given by,

$$F_A = R_A - 2x \qquad (\because\ R_A = 3)$$
$$= 3 - 2x \qquad ...(i)$$

At A, $x = 0$ hence $F_A = 3$ kN

At B, $x = 4$ hence $F_B = 3 - 2 \times 4 = -5$ kN

The shear force varies according to straight line law between A and B. At A, the shear force is positive whereas at B, the shear force is negative. Between A and B somewhere S.F. is zero. The point, where S.F. is zero, is obtained by substituting $F_x = 0$ in equation (*i*).

$$\therefore \quad 0 = 3 - 2x \quad \text{or} \quad x = \frac{3}{2} = 1.5 \text{ m}$$

Hence S.F. is zero at a distance of 1.5 m from A (or S.F. is zero at point D).

(*ii*) The S.F. at any section between B and C at a distance x from A is given by,

$$F_x = +R_A - 4 \times 2 + R_B - (x-4) \times 2 = 3 - 8 + 9 - 2(x-4)$$
$$= 4 - 2(x-4) \qquad ...(ii)$$

At B, $x = 4$ m hence $F_B = 4 - 2(4-4) = +4$ kN

At C, $x = 6$ m hence $F_C = 4 - 2(6-4) = 0$

Between B and C also S.F. varies by a straight line law. At B, S.F. is + 4 kN and at C, S.F. is zero.

The S.F. diagram is shown in Fig. 6.35 (*b*).

Bending Moment Diagram

The B.M. at A is zero.

(*i*)The B.M. at any section between A and B at a distance x is given by,

$$M_x = R_A \times x - 2 \times x \times \frac{x}{2}$$
$$= 3x - x^2 \qquad ...(iii)$$

At A, $x = 0$ hence $M_A = 0$

At B, $x = 4$ hence $M_B = 3 \times 4 - 4^2 = -4$ kNm

Max. B.M. occurs at D, where S.F. is zero after changing its sign.

At D, $x = 1.5$ hence $M_D = 3 \times 1.5 - 1.5 = 4.5 - 2.25 = 2.25$ kNm

The B.M. between A and B varies according to parabolic law.

(*ii*)The B.M. at any section between B and C at a distance x is given by,

$$M_x = R_A \times x - 2 \times x \times \frac{x}{2} + R_B \times (x-4)$$
$$= 3x - x^2 + 9(x-4) \qquad ...(iv)$$

At B, $x = 4$ hence $M_B = 3 \times 4 - 4^2 + 9(4-4) = 4$ kNm

At C, $x = 6$ hence $M_C = 3 \times 6 - 6^2 + 9(6-4) = 18 - 36 + 18 = 0$

The B.M. diagram is shown in Fig. 6.35 (*c*).

Point of Contraflexure

This point will be between A and B where B.M. is zero after changing its sign. But B.M. at any section at a distance x from A between A and B is given by equation (*iii*) as

$$M_x = 3x - x^2$$

Equation M_x to zero for point of contraflexure, we get

$$0 = 3x - x^2 = x(3 - x)$$

or $\quad 3 - x = 0 \qquad$ ($\because$ x cannot be zero as B.M. is not changing sign at this point)

$$\therefore \quad x = 3$$

Hence point of contraflexure will be at a distance of 3 m from A.

LECTURER NOTES

3RD SEMESTER

MECHANICAL ENGINEERING

2022

Strength of Material

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Chapter- 05

THEORY OF SIMPLE BENDING

When some external load acts on a beam, the shear force and bending moments are set up at all sections of the beam. Due to the shear force and bending moment, the beam undergoes certain deformation. The material of the beam will offer resistance or stresses against these deformations. These stresses with certain assumptions can be calculated. The stresses introduced by bending moment are known as *bending stresses*. In this chapter, the theory of pure bending, expression for bending stresses, bending stress in symmetrical and unsymmetrical sections, strength of a beam and composite beams will be discussed.

PURE BENDING OR SIMPLE BENDING

If a length of a beam is subjected to a constant bending moment and no shear force (*i.e.*, zero shear force), then the stresses will be set up in that length of the beam due to B.M. only and that length of the beam is said to be in *pure bending* or simple bending. The stresses set up in that length of beam are known as bending stresses.

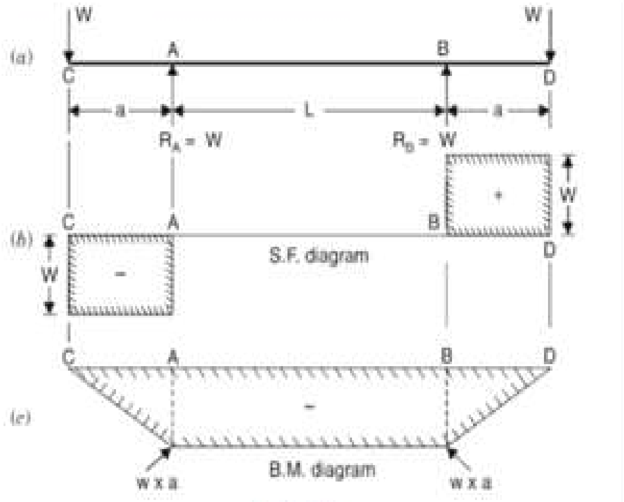


Fig. 7.1

A beam simply supported at *A* and *B* and overhanging by same length at each support is shown in Fig. 7.1. A point load *W* is applied at each end of the overhanging portion. The

S.F. and B.M. for the beam are drawn as shown in Fig. 7.1 (*b*) and Fig. 7.1 (*c*) respectively. From these diagrams, it is clear that there is no shear force between *A* and *B* but the B.M. between *A* and *B* is constant.

This means that between *A* and *B*, the beam is subjected to a constant bending moment only. This condition of the beam between *A* and *B* is known as pure bending or simple bending.

ASSUMPTIONS OF THEORY OF SIMPLE BENDING

Before discussing the theory of simple bending, let us see the assumptions made in the theory of simple bending. The following are the important assumptions :

1. The material of the beam is homogeneous* and isotropic**.

2. The value of Young's modulus of elasticity is the same in tension and compression.

3. The transverse sections which were plane before bending, remain plane after bending also.

4. The beam is initially straight and all longitudinal filaments bend into circular arcs with a common centre of curvature.

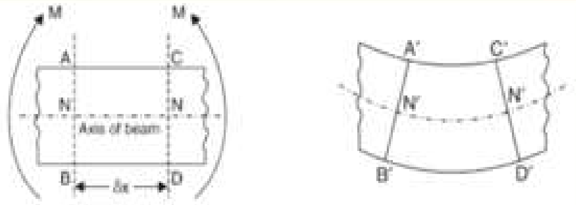
5. The radius of curvature is large compared with the dimensions of the cross-section.

6. Each layer of the beam is free to expand or contract, independently of the layer, above or below it.

THEORY OF SIMPLE BENDING

Fig. 7.2 (*a*) shows a part of a beam subjected to simple bending. Consider a small length δx of this part of beam. Consider two sections *AB* and *CD* which are normal to the axis of the beam *N – N*. Due to the action of the bending moment, the part of length δx will be deformed as shown in Fig. 7.2 (*b*). From this figure, it is clear that all the layers of the beam, which were originally of the same length, do not remain of the same length any more.

The top layer such as *AC* has deformed to the shape *A′C′*. This layer has been shortened in its length. The bottom layer *BD* has deformed to the shape *B′D′*. This layer has been elongated. From the Fig. 7.2 (*b*), it is clear that some of the layers have been shortened while some of them are elongated. At a level between the top and bottom of the beam, there will be a layer which is neither shortened nor elongated. This layer is known as *neutral layer* or *neutral*



(*a*) Before bending (*b*) After bending

Fig. 7.2

surface. This layer in Fig. 7.2 (*b*) is shown by *N′ – N′* and in Fig. 7.2 (*a*) by *N – N*. The line of intersection of the neutral layer on a cross-section of a beam is known as *neutral axis* (written as N.A.).

The layers above *N – N* (or *N′ – N′*) have been shortened and those below, have been elongated. Due to the decrease in lengths of the layers above *N – N*, these layers will be subjected to compressive stresses. Due to the increase in the lengths of layers below *N – N*, these layers will be subjected to tensile stresses.

We also see that the top layer has been shortened maximum. As we proceed towards the layer $N-N$, the decrease in length of the layers decreases. At the layer $N-N$, there is no change in length. This means the compressive stress will be maximum at the top layer. Similarly the increase in length will be maximum at the bottom layer. As we proceed from bottom layer towards the layer $N-N$, the increase in length of layers decreases. Hence the amount by which a layer increases or decreases in length, depends upon the position of the layer with respect to $N-N$. This theory of bending is known as theory of simple bending.

EXPRESSION FOR BENDING STRESS

Fig. 7.3 (*a*) shows a small length δx of a beam subjected to a simple bending. Due to the action of bending, the part of length δx will be deformed as shown in Fig. 7.3 (*b*). Let $A'B'$ and $C'D'$ meet at O.

Let R = Radius of neutral layer $N'N'$

θ = Angle subtended at O by $A'B'$ and $C'D'$ produced.

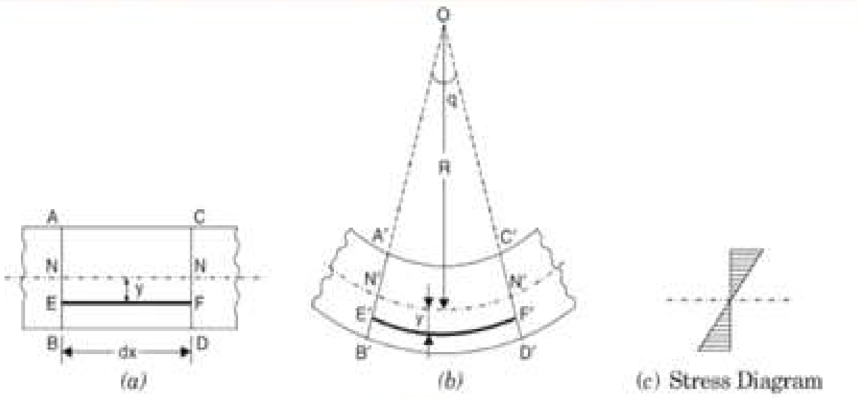


Fig. 7.3

7.4.1. Strain Variation Along the Depth of Beam. Consider a layer EF at a distance y below the neutral layer NN. After bending this layer will be elongated to $E'F'$.

Original length of layer $EF = \delta x$.

Also length of neutral layer $NN = \delta x$.

After bending, the length of neutral layer $N'N'$ will remain unchanged. But length of layer $E'F'$ will increase. Hence

$$N'N' = NN = \delta x.$$

Now from Fig. 7.3 (*b*),

$$N'N' = R \times \theta$$

and $$E'F' = (R + y) \times \theta \qquad (\because \text{ Radius of } E'F' = R + y)$$

But $$N'N' = NN = \delta x.$$

Hence $$\delta x = R \times \theta$$

$\therefore$ Increase in the length of the layer EF

$$= E'F' - EF = (R + y)\,\theta - R \times \theta \qquad (\because EF = \delta x = R \times \theta)$$

$$= y \times \theta$$

∴ Strain in the layer EF

$$= \frac{\text{Increase in length}}{\text{Original length}}$$

$$= \frac{y \times \theta}{EF} = \frac{y \times \theta}{R \times \theta} \qquad (\because EF = \delta x = R \times \theta)$$

$$= \frac{y}{R}$$

As R is constant, *hence the strain in a layer is proportional to its distance from the neutral axis.* The above equation shows the variation of strain along the depth of the beam. The variation of strain is linear.

STRESS VARIATION

Let σ = Stress in the layer EF

E = Young's modulus of the beam

Then $$E = \frac{\text{Stress in the layer } EF}{\text{Strain in the layer } EF}$$

$$= \frac{\sigma}{\left(\frac{y}{R}\right)} \qquad \left(\because \text{ Strain in } EF = \frac{y}{R}\right)$$

∴ $$\sigma = E \times \frac{y}{R} = \frac{E}{R} \times y \qquad ...(7.1)$$

Since E and R are constant, *therefore stress in any layer is directly proportional to the distance of the layer from the neutral layer.* The equation (7.1) shows the variation of stress along the depth of the beam. The variation of stress is linear.

In the above case, all layers below the neutral layer are subjected to tensile stresses whereas the layers above neutral layer are subjected to compressive stresses. The Fig. 7.3 (c) shows the stress distribution.

Equation (7.1) can also be written as

$$\frac{\sigma}{y} = \frac{E}{R} \qquad ...(7.2)$$

NEUTRAL AXIS AND MOMENT OF RESISTANCE

The neutral axis of any transverse section of a beam is defined as the line of intersection of the neutral layer with the transverse section. It is written as N.A.

In Art. 7.4, we have seen that if a section of a beam is subjected to pure sagging moment, then the stresses will be compressive at any point above the neutral axis and tensile below the neutral axis. There is no stress at the neutral axis. The stress at a distance y from the neutral axis is given by equation (7.1) as

$$\sigma = \frac{E}{R} \times y.$$

Fig. 7.4 shows the cross-section of a beam. Let N.A. be the neutral axis of the section. Consider a small layer at a distance y from the neutral axis. Let dA = Area of the layer.

Now the force on the layer

= Stress on layer × Area of layer

$= \sigma \times dA$

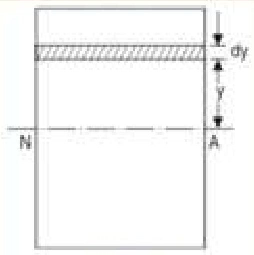


Fig. 7.4

$$= \frac{E}{R} \times y \times dA \qquad ...(i) \quad \left(\because \ \sigma = \frac{E}{R} \times y\right)$$

Total force on the beam section is obtained by integrating the above equation.

∴ Total force on the beam section

$$= \int \frac{E}{R} \times y \times dA$$

$$= \frac{E}{R} \int y \times dA \qquad (\because \ E \text{ and } R \text{ is constant})$$

But for pure bending, there is no force on the section of the beam (or force is zero).

$$\therefore \qquad \frac{E}{R} \int y \times dA = 0$$

or

$$\int y \times dA = 0 \qquad \left(\text{as } \frac{E}{R} \text{ cannot be zero}\right)$$

Now $y \times dA$ represents the moment of area dA about neutral axis. Hence $\int y \times dA$ represents the moment of entire area of the section about neutral axis. But we know that moment of any area about an axis passing through its centroid, is also equal to zero. Hence neutral axis coincides with the centroidal axis. Thus the centroidal axis of a section gives the position of neutral axis.

Moment of Resistance. Due to pure bending, the layers above the N.A. are subjected to compressive stresses whereas the layers below the N.A. are subjected to tensile stresses. Due to these stresses, the forces will be acting on the layers. These forces will have moment about the N.A. The total moment of these forces about the N.A. for a section is known as moment of resistance of that section.

The force on the layer at a distance y from neutral axis in Fig. 7.4 is given by equation (*i*), as

$$\text{Force on layer} = \frac{E}{R} \times y \times dA$$

Moment of this force about N.A.

$$= \text{Force on layer} \times y$$

$$= \frac{E}{R} \times y \times dA \times y$$

$$= \frac{E}{R} \times y^2 \times dA$$

Total moment of the forces on the section of the beam (or moment of resistance)

$$= \int \frac{E}{R} \times y^2 \times dA = \frac{E}{R} \int y^2 \times dA$$

Let M = External moment applied on the beam section. For equilibrium the moment of resistance offered by the section should be equal to the external bending moment.

$$\therefore \qquad M = \frac{E}{R} \int y^2 \times dA.$$

But the expression $\int y^2 \times dA$ represents the moment of inertia of the area of the section about the neutral axis. Let this moment of inertia be I.

$$\therefore \qquad M = \frac{E}{R} \times I \quad \text{or} \quad \frac{M}{I} = \frac{E}{R} \qquad ...(7.3)$$

But from equation (7.2), we have

$$\frac{\sigma}{y} = \frac{E}{R}$$

$$\therefore \qquad \frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R} \qquad ...(7.4)$$

Equation (7.4) is known as bending equation.

In equation (7.4), the different quantities are expressed in consistent units as given below :

M is expressed in N mm ; I in mm^4

σ is expressed in N/mm^2 ; y in mm

and E is expressed in N/mm^2 ; R in mm.

Condition of Simple Bending. Equation (7.4) is applicable to a member which is

subjected to a constant bending moment and the member is absolutely free from shear force. But in actual practice, a member is subjected to such loading that the B.M. varies from section to section and also the shear force is not zero. But shear force is zero at a section where bending moment is maximum. *Hence the condition of simple bending* may be assumed to be satisfied at such a section. Hence the stresses produced due to maximum bending moment, are obtained from equation (7.4) as the shear forces at these sections are generally zero. Hence the theory and equations discussed in the above articles are quite sufficient and give results which enables the engineers to design beams and structures and calculate their stresses and strains with a reasonable degree of approximation where B.M. is maximum.

BENDING STRESSES IN SYMMETRICAL SECTION

The neutral axis (N.A.) of a symmetrical section (such as circular, rectangular or square) lies at a distance of $d/2$ from the outermost layer of the section where d is the diameter (for a circular section) or depth (for a rectangular or a square section). There is no stress at the neutral axis. But the stress at a point is directly proportional to its distance from the neutral axis. The maximum stress takes place at the outermost layer. For a simply supported beam, there is a compressive stress above the neutral axis and a tensile stress below it. If we plot these stresses, we will get a figure as shown in Fig. 7.5.

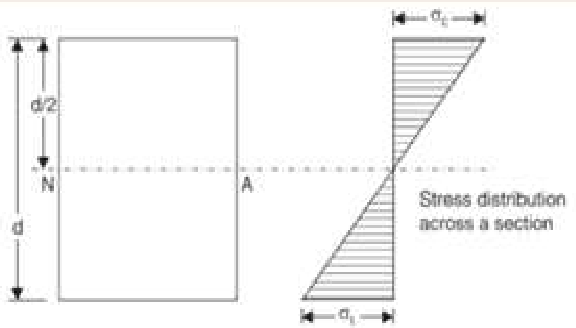


Fig. 7.5

Problem 7.1. *A steel plate of width 120 mm and of thickness 20 mm is bent into a circular arc of radius 10 m. Determine the maximum stress induced and the bending moment which will produce the maximum stress. Take $E = 2 \times 10^5$ N/mm².*

Sol. Given :

Width of plate, $b = 120$ mm

Thickness of plate, $t = 20$ mm

$\therefore$ Moment of inertia, $I = \frac{bt^3}{12} = \frac{120 \times 20^3}{12} = 8 \times 10^4$ mm⁴

Radius of curvature, $R = 10$ m $= 10 \times 10^3$ mm

Young's modulus, $E = 2 \times 10^5$ N/mm²

Let σ_{max} = Maximum stress induced, and

M = Bending moment.

Using equation (7.2), $\frac{\sigma}{y} = \frac{E}{R}$

$$\therefore \quad \sigma = \frac{E}{R} \times y \qquad ...(i)$$

Equation (*i*) gives the stress at a distance *y* from N.A.

Stress will be maximum, when *y* is maximum. But *y* will be maximum at the top layer or bottom layer.

$$\therefore \quad y_{max} = \frac{t}{2} = \frac{20}{2} = 10 \text{ mm.}$$

Now equation (*i*) can be written as

$$\sigma_{max} = \frac{E}{R} \times y_{max}$$

$$= \frac{2 \times 10^5}{10 \times 10^3} \times 10 = \mathbf{200 \text{ N/mm}^2}. \quad \textbf{Ans.}$$

From equation (7.4), we have

$$\frac{M}{I} = \frac{E}{R}$$

$$\therefore \quad M = \frac{E}{R} \times I = \frac{2 \times 10^5}{10 \times 10^3} \times 8 \times 10^4$$

$$= 16 \times 10^5 \text{ N mm} = \mathbf{1.6 \text{ kNm}}. \quad \textbf{Ans.}$$

Problem 7.2. *Calculate the maximum stress* induced in a cast iron pipe of external diameter 40 mm, of internal diameter 20 mm and of length 4 metre when the pipe is supported at its ends and carries a point load of 80 N at its centre.*

Sol. Given :

External dia., $D = 40$ mm

Internal dia., $d = 20$ mm

Length, $L = 4$ m $= 4 \times 1000 = 4000$ mm

Point load, $W = 80$ N

In case of simply supported beam carrying a point load at the centre, the maximum bending moment is at the centre of the beam.

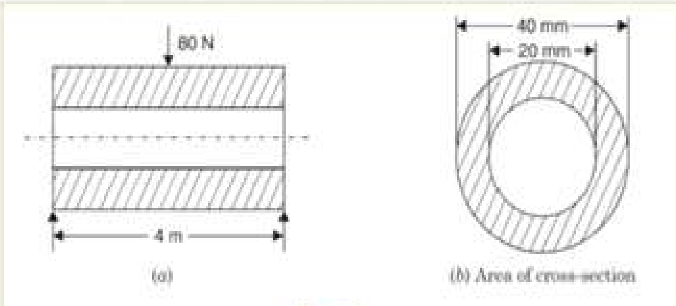


Fig. 7.6

And maximum B.M. $= \dfrac{W \times L}{4}$

$\therefore$ Maximum B.M. $= \dfrac{80 \times 4000}{4} = 8 \times 10^4$ Nmm

$\therefore$ $M = 8 \times 10^4$ Nmm

Fig. 7.6 (*b*) shows the cross-section of the pipe.

Moment of inertia of hollow pipe,

$$I = \frac{\pi}{64}\,[D^4 - d^4]$$

$$= \frac{\pi}{64}\,[40^4 - 20^4] = \frac{\pi}{64}\,[2560000 - 160000]$$

$$= 117809.7 \text{ mm}^4$$

Now using equation (7.4),

$$\frac{M}{I} = \frac{\sigma}{y} \qquad \ldots(i)$$

when y is maximum, stress will be maximum. But y is maximum at the top layer from the N.A.

$\therefore$ $y_{max} = \dfrac{D}{2} = \dfrac{40}{2} = 20$ mm

Equation (*i*) can be written as

$$\frac{M}{I} = \frac{\sigma_{max}}{y_{max}}$$

$\therefore$ $\sigma_{max} = \dfrac{M}{I} \times y_{max}$

$$= \frac{8 \times 10^4 \times 20}{117809.7} = \mathbf{13.58\ N/mm^2}. \quad \textbf{Ans.}$$

SECTION MODULUS

Section modulus is defined as the ratio of moment of inertia of a section about the neutral axis to the distance of the outermost layer from the neutral axis. It is denoted by the symbol Z. Hence mathematically section modulus is given by,

$$Z = \frac{I}{y_{max}} \qquad \ldots(7.5)$$

where I = M.O.I. about neutral axis

and y_{max} = Distance of the outermost layer from the neutral axis.

From equation (7.4), we have

$$\frac{M}{I} = \frac{\sigma}{y}$$

The stress σ will be maximum, when y is maximum. Hence above equation can be written as

$$\frac{M}{I} = \frac{\sigma_{max}}{y_{max}}$$

$$\therefore \quad M = \sigma_{max} \cdot \frac{I}{y_{max}}$$

But $\frac{I}{y_{max}} = Z$

$$\therefore \quad M = \sigma_{max} \cdot Z \qquad \text{...(7.6)}$$

In the above equation, M is the maximum bending moment (or moment of resistance offered by the section). Hence moment of resistance offered the section is maximum when section modulus Z is maximum. Hence section modulus represent the strength of the section.

SECTION MODULUS FOR VARIOUS SHAPES OR BEAM SECTION

1. *Rectangular Section*

Moment of inertia of a rectangular section about an axis through its C.G. (or through N.A.) is given by,

$$I = \frac{bd^3}{12}$$

Distance of outermost layer from N.A. is given by,

$$y_{max} = \frac{d}{2}$$

∴ Section modulus is given by,

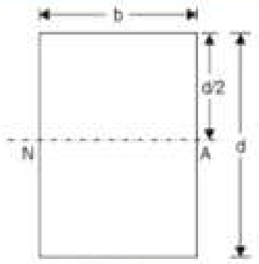


Fig. 7.7

$$Z = \frac{I}{y_{max}} = \frac{bd^3}{12 \times \left(\frac{d}{2}\right)} = \frac{bd^3}{12} \times \frac{2}{d} = \frac{bd^2}{6} \qquad \text{...(7.7)}$$

2. *Hollow Rectangular Section*

Here $I = \frac{BD^3}{12} - \frac{bd^2}{12}$

$$= \frac{1}{12}[BD^3 - bd^3]$$

$$y_{max} = \frac{D}{2}$$

$$\therefore \quad Z = \frac{I}{y_{max}}$$

$$= \frac{\frac{1}{12}[BD^3 - bd^3]}{\left(\frac{D}{2}\right)}$$

$$= \frac{1}{6D}[BD^3 - bd^3] \qquad \text{...(7.8)}$$

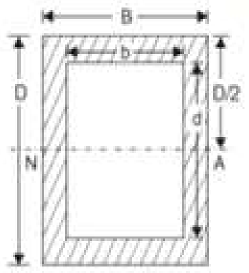


Fig. 7.8

3. *Circular Section*

For a circular section,

$$I = \frac{\pi}{64} d^4 \quad \text{and} \quad y_{max} = \frac{d}{2}$$

$$\therefore \quad Z = \frac{I}{y_{max}} = \frac{\frac{\pi}{64} d^4}{\left(\frac{d}{2}\right)} = \frac{\pi}{32} d^3 \qquad \text{...(7.9)}$$

4. *Hollow Circular Section*

Here $I = \frac{\pi}{64}[D^4 - d^4]$

and $y_{max} = \frac{D}{2}$

$$\therefore \quad Z = \frac{I}{y_{max}} = \frac{\frac{\pi}{64}[D^4 - d^4]}{\left(\frac{D}{2}\right)}$$

$$= \frac{\pi}{32D}[D^4 - d^4] \quad ...(7.10)$$

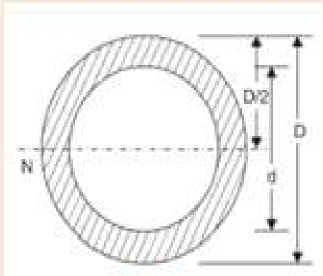


Fig. 7.9

Problem 7.3. *A cantilever of length 2 metre fails when a load of 2 kN is applied at the free end. If the section of the beam is 40 mm × 60 mm, find the stress at the failure.*

Sol. Given :

Length, $L = 2 \text{ m} = 2 \times 10^3$ mm

Load, $W = 2 \text{ kN} = 2000$ N

Section of beam is 40 mm × 60 mm.

∴ Width of beam, $b = 40$ mm

Depth of beam, $d = 60$ mm

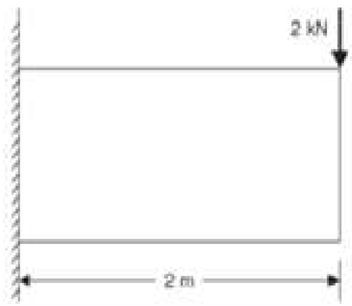


Fig. 7.10

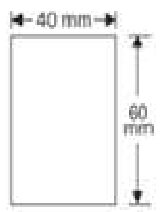


Fig. 7.10 (*a*)

Fig. 7.10 (*a*) shows the section of the beam.

Section modulus of a rectangular section is given by equation (7.7).

$$\therefore \quad Z = \frac{bd^2}{6} = \frac{40 \times 60^2}{6} = 24000 \text{ mm}^3$$

Maximum bending moment for a cantilever shown in Fig. 7.10 is at the fixed end.

$$\therefore \quad M = W \times L = 2000 \times 2 \times 10^3 = 4 \times 10^6 \text{ Nmm}$$

Let σ_{max} = Stress at the failure

Using equation (7.6), we get

$$M = \sigma_{max} \cdot Z$$

$$\therefore \quad \sigma_{max} = \frac{M}{Z} = \frac{4 \times 10^6}{24000} = \mathbf{166.67 \text{ N/mm}^2}. \quad \textbf{Ans.}$$

Problem 7.4. *A rectangular beam 200 mm deep and 300 mm wide is simply supported over a span of 8 m. What uniformly distributed load per metre the beam may carry, if the bending stress is not to exceed 120 N/mm².*

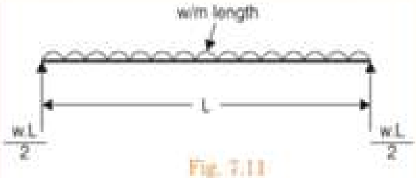


Fig. 7.11

Sol. Given :

Depth of beam, $d = 200$ mm

Width of beam, $b = 300$ mm

Length of beam, $L = 8$ m

Max. bending stress,

$\sigma_{max} = 120 \text{ N/mm}^2$

Let w = Uniformly distributed load per metre length over the beam

(Fig. 7.11 (*a*) shows the section of the beam.)

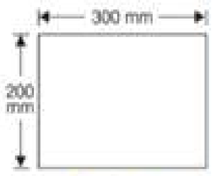


Fig. 7.11 (*a*)

Section modulus for a rectangular section is given by equation (7.7).

$$\therefore \quad Z = \frac{bd^2}{6} = \frac{300 \times 200^2}{6} = 2000000 \text{ mm}^3$$

Max. B.M. for a simply supported beam carrying uniformly distributed load as shown in Fig. 7.11 is at the centre of the beam. It is given by

$$M = \frac{w \times L^2}{8} = \frac{w \times 8^2}{8} \quad (\because \; L = 8 \text{ m})$$

$$= 8w \text{ Nm} = 8w \times 1000 \text{ Nmm}$$

$$= 8000w \text{ Nmm} \quad (\because \; 1 \text{ m} = 1000 \text{ mm})$$

Now using equation (7.6), we get

$$M = \sigma_{max.} \cdot Z$$

or $$8000w = 120 \times 2000000$$

$$\therefore \quad w = \frac{120 \times 2000000}{8000} = 30 \times 1000 \text{ N/m} = \mathbf{30 \text{ kN/m.}} \quad \textbf{Ans.}$$

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LECTURER NOTES

3RD SEMESTER

MECHANICAL ENGINEERING

2022

Strength of Material

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Chapter- 06

COMBINED DIRECT AND BENDING STRESSES

Direct stress alone is produced in a body when it is subjected to an axial tensile or compressive load. And bending stress is produced in the body, when it is subjected to a bending moment. But if a body is subjected to axial loads and also bending moments, then both the stresses (*i.e.*, direct and bending stresses) will be produced in the body. In this chapter, we shall study the important cases of the members subjected to direct and bending stresses. Both these stresses act normal to a cross-section, hence the two stresses may be algebraically added into a single resultant stress.

COLUMN

Column or strut is defined as a member of a structure, which is subjected to axial compressive load. If the member of the structure is vertical and both of its ends are fixed rigidly while subjected to axial compressive load, the member is known as *column*, for example a vertical pillar between the roof and floor. If the member of the structure is not vertical and one or both of its ends are hinged or pin joined, the bar is known as *strut*. Examples of struts are : connecting rods, piston rods etc.

COMBINED DIRECT AND BENDING STRESS

Consider the case of a column* subjected by a compressive load P acting along the axis of the column as shown in Fig. 9.1. This load will cause a direct compressive stress whose intensity will be uniform across the cross-section of the column.

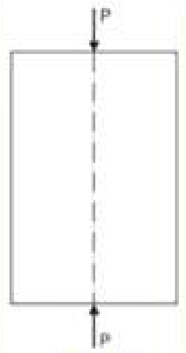

Fig. 9.1

Let σ_0 = Intensity of the stress

A = Area of cross-section

P = Load acting on the column.

Then stress,

$$\sigma_0 = \frac{\text{Load}}{\text{Area}} = \frac{P}{A}$$

Now consider the case of a column subjected by a compressive load P whose line of action is at a distance of 'e' from the axis of the column as shown in Fig. 9.2 (a). Here 'e' is known as eccentricity of the load. The eccentric load shown in Fig. 9.2 (a) will cause direct stress and bending stress. This is proved as discussed below :

1. In Fig. 9.2 (b), we have applied, along the axis of the column, two equal and opposite forces P. Thus three forces are acting now on the column. One of the forces is shown in Fig. 9.2 (c) and the other two forces are shown in Fig. 9.2 (d).

2. The force shown in Fig. 9.2 (c) is acting along the axis of the column and hence this force will produce a direct stress.

3. The forces shown in Fig. 9.2 (d) will form a couple, whose moment will be $P \times e$. This couple will produce a bending stress.

Hence an eccentric* load will produce a direct stress as well as a bending stress. By adding these two stresses algebraically, a single resultant stress can be obtained.

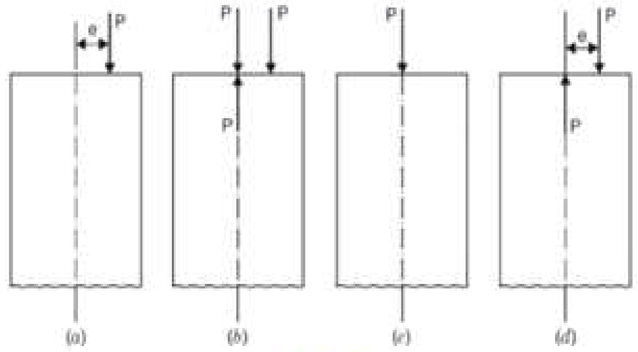

Fig. 9.2

RESULTANT STRESS WHEN A COLUMN OF RECTANGULAR SECTION IS SUBJECTED TO AN ECCENTRIC LOAD

A column of rectangular section subjected to an eccentric load is shown in Fig. 9.3. Let the load is eccentric with respect to the axis *Y-Y* as shown in Fig. 9.3 (*b*). It is mentioned in Art. 9.2 that an eccentric load causes direct stress as well as bending stress. Let us calculate these stresses.

Let P = Eccentric load on column

e = Eccentricity of the load

σ_0 = Direct stress

σ_b = Bending stress

b = Width of column

d = Depth of column

∴ Area of column section, $A = b \times d$

Now moment due to eccentric load P is given by,

$$M = \text{Load} \times \text{eccentricity}$$
$$= P \times e$$

The direct stress (σ_0) is given by,

$$\sigma_0 = \frac{\text{Load } (P)}{\text{Area}} = \frac{P}{A} \quad ...(i)$$

This stress is uniform along the cross-section of the column.

The bending stress σ_b due to moment at any point of the column section at a distance y from the neutral axis *Y-Y* is given by

$$\frac{M}{I} = \frac{\sigma_b}{\pm y}$$

$$\therefore \qquad \sigma_b = \pm \frac{M}{I} \times y \qquad \ldots(ii)$$

where I = Moment of inertia of the column section about the neutral axis Y-$Y = \frac{d.b^3}{12}$

Substituting the value of I in equation (ii), we get

$$\sigma_b = \pm \frac{M}{\frac{d.b^3}{12}} \times y = \pm \frac{12\,M}{d.b^3} \times y$$

The bending stress depends upon the value of y from the axis Y-Y.

The bending stress at the extreme is obtained by substituting $y = \frac{b}{2}$ in the above equation.

$$\therefore \qquad \sigma_b = \pm \frac{12\,M}{d.b^3} \times \frac{b}{2} = \pm \frac{6\,M}{d.b^2}$$

$$= \pm \frac{6\,P \times e}{d.b^2} \qquad (\because \; M = P \times e)$$

$$= \pm \frac{6\,P \times e}{d.b.b} = \pm \frac{6\,P \times e}{A \times b}$$

$$(\because \; \text{Area} = b \times d = A)$$

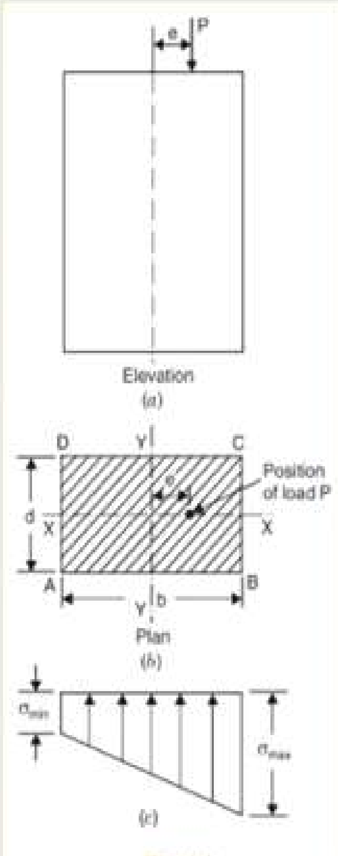

Fig. 9.3

The resultant stress at any point will be the algebraic sum of direct stress and bending stress.

If y is taken positive on the same side of Y-Y as the load, then bending stress will be of the same type as the direct stress. Here direct stress is compressive and hence bending stress will also be compressive towards the right of the axis Y-Y. Similarly bending stress will be tensile towards the left of the axis Y-Y. Taking compressive stress as positive and tensile stress as negative we can find the maximum and minimum stress at the extremities of the section. The stress will be maximum along layer BC and minimum along layer AD.

Let σ_{max} = Maximum stress (*i.e.*, stress along BC)

σ_{min} = Minimum stress (*i.e.*, stress along AD)

Then σ_{max} = Direct stress + Bending stress

$$= \sigma_0 + \sigma_b$$

$$= \frac{P}{A} + \frac{6\,P.e}{A.b} \qquad \text{(Here bending stress is +ve)}$$

$$= \frac{P}{A}\left(1 + \frac{6 \times e}{b}\right) \qquad \ldots(9.1)$$

and σ_{min} = Direct stress – Bending stress

$$= \sigma_0 - \sigma_b$$

$$= \frac{P}{A} - \frac{6\,P.e}{A.b} = \frac{P}{A}\left(1 - \frac{6 \times e}{b}\right) \qquad \ldots(9.2)$$

These stresses are shown in Fig. 9.3 (*c*). The resultant stress along the width of the column will vary by a straight line law.

If in equation (9.2), σ_{min} is negative then the stress along the layer *AD* will be tensile. If σ_{min} is zero then there will be no tensile stress along the width of the column. If σ_{min} is positive then there will be only compressive stress along the width of the column.

Problem 9.1. *A rectangular column of width 200 mm and of thickness 150 mm carries a point load of 240 kN at an eccentricity of 10 mm as shown in Fig. 9.4 (i). Determine the maximum and minimum stresses on the section.*

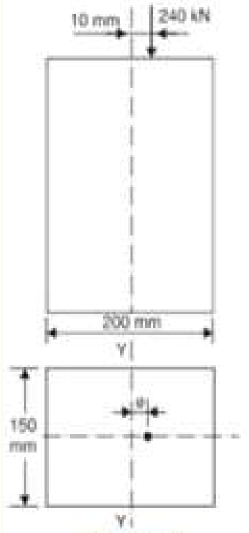

Fig. 9.4 (*i*)

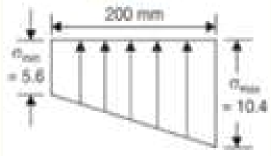

Fig. 9.4 (*ii*)

Sol. Given :

Width, $b = 200$ mm

Thickness, $d = 150$ mm

∴ Area, $A = b \times d$
$= 200 \times 150 = 30000 \text{ mm}^2$

Eccentric load,
$P = 240$ kN
$= 240000$ N

Eccentricity,
$e = 10$ mm

Let σ_{max} = Maximum stress, and
σ_{min} = Minimum stress.

(*i*) Using equation (9.1), we get

$$\sigma_{max} = \frac{P}{A}\left(1 + \frac{6 \times e}{b}\right)$$

$$= \frac{240000}{30000}\left(1 + \frac{6 \times 10}{200}\right)$$

$$= 8(1 + 0.3) = \mathbf{10.4 \ N/m^2}. \quad \textbf{Ans.}$$

(*ii*) Using equation (9.2), we get

$$\sigma_{min} = \frac{P}{A}\left(1 - \frac{6 \times e}{b}\right)$$

$$= \frac{240000}{30000}\left(1 - \frac{6 \times 10}{200}\right) = 8(1 - 0.3) = \mathbf{5.6 \ N/mm^2}. \quad \textbf{Ans.}$$

These stresses are shown in Fig. 9.4 (*ii*).

RESULTANT STRESS WHEN A COLUMN OF RECTANGULAR SECTION IS SUBJECTED TO A LOAD WHICH IS ECCENTRIC TO BOTH AXES

A column of rectangular section *ABCD*, subjected to a load which is eccentric to both axes, is shown in Fig. 9.11.

Let P = Eccentric load on column

e_x = Eccentricity of load about *X-X* axis

e_y = Eccentricity of load about *Y-Y* axis

b = Width of column

d = Depth of column

σ_0 = Direct stress

σ_{bx} = Bending stress due to eccentricity e_x

σ_{by} = Bending stress due to eccentricity e_y

M_x = Moment of load about *X-X* axis $= P \times e_x$

M_y = Moment of load about *Y-Y* axis $= P \times e_y$

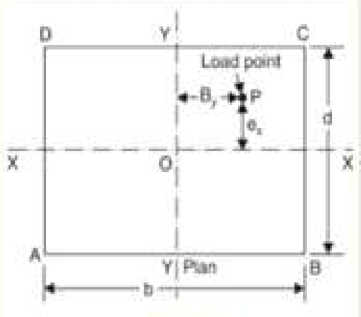

Fig. 9.11

I_{xx} = Moment of inertia about *X-X* axis

$$= \frac{bd^3}{12}$$

I_{yy} = Moment of inertia about *Y-Y* axis

$$= \frac{db^3}{12}$$

Now the eccentric load is equivalent to a central load P, together with a bending moment $P \times e_y$ about *Y-Y* and a bending moment $P \times e_x$ about *X-X*.

(*i*) The direct stress (σ_0) is given by,

$$\sigma_0 = \frac{P}{A} \qquad \text{...}(i)$$

(*ii*) The bending stress due to eccentricity e_y is given by,

$$\sigma_{by} = \frac{M_y \times x}{I_{yy}} = \frac{P \times e_y \times x}{I_{yy}} \qquad (\because \; M_y = B \times e_y) \quad \text{...}(ii)$$

In the above equation x varies from $-\frac{b}{2}$ to $+\frac{b}{2}$

(*iii*) The bending stress due to eccentricity e_x is given by,

$$\sigma_{bx} = \frac{M_x \times y}{I_{xx}} = \frac{P \times e_x \times y}{I_{xx}}$$

In the above equation, y varies from $-\frac{d}{2}$ to $+\frac{d}{2}$

The resultant stress at any point on the section

$$= \sigma_0 \pm \sigma_{by} \pm \sigma_{bx}$$

$$= \frac{P}{A} \pm \frac{M_y \times x}{I_{yy}} \pm \frac{M_x \cdot y}{I_{xx}} \qquad \text{...}(9.3)$$

(*i*) At the point *C*, the co-ordinates *x* and *y* are positive hence the resultant stress will be maximum.

(*ii*) At the point *A*, the co-ordinates *x* and *y* are negative and hence the resultant stress will be minimum.

(*iii*) At the point *B*, *x* is +ve and *y* is –ve and hence resultant stress

$$= \frac{P}{A} + \frac{M_y \cdot x}{I_{yy}} - \frac{M_x \cdot y}{I_{xx}}$$

(*iv*) At the point D, x is –ve and y is +ve and hence resultant stress

$$= \frac{P}{A} - \frac{M_y \cdot x}{I_{yy}} + \frac{M_x \cdot y}{I_{xx}}.$$

FAILURE OF A COLUMN

The failure of a column takes place due to anyone of the following stresses set up in the columns :

(*i*) Direct compressive stresses,

(*ii*) Buckling stresses, and

(*iii*) Combined direct compressive and buckling stresses.

19.2.1. Failure of a Short Column. A short column of uniform cross-sectional area A, subjected to an axial compressive load P, is shown in Fig. 19.1. The compressive stress induced is given by

$$p = \frac{P}{A}$$

If the compressive load on the short column is gradually increased, a stage will reach when the column will be on the point of failure by crushing. The stress induced in the column corresponding to this load is known as crushing stress and the load is called crushing load.

Let P_c = Crushing load,

σ_c = Crushing stress, and

A = Area of cross-section.

Then $$\sigma_c = \frac{P_c}{A}.$$

All short columns fail due to crushing.

Fig. 19.1

19.2.2. Failure of a Long Column. A long column of uniform cross-sectional area A and of length l, subjected to an axial compressive load P, is shown in Fig. 19.2. A column is known as long column if the length of the column in comparison to its lateral dimensions, is very large. Such columns do not fail by crushing alone, but also by bending (also known buckling) as shown in Fig. 19.2. The load at which the column just buckles, is known as *buckling load or critical just or crippling load*. The buckling load is less than the crushing load for a long column. Actually the value of buckling load for long columns is low whereas for short columns the value of buckling load is relatively high.

Refer to Fig. 19.2.

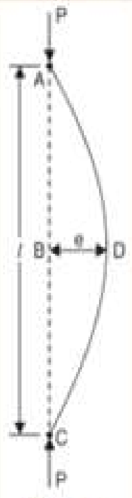

Fig. 19.2

Let l = Length of a long column

P = Load (compressive) at which the column has just buckled

A = Cross-sectional area of the column

e = Maximum bending of the column at the centre

σ_0 = Stress due to direct load = $\frac{P}{A}$

σ_b = Stress due to bending at the centre of the column = $\frac{P \times e}{Z}$

where Z = Section modulus about the axis of bending.

The extreme stresses on the mid-section are given by

Maximum stress = $\sigma_0 + \sigma_b$

and Minimum stress = $\sigma_0 - \sigma_b$.

The column will fail when maximum stress (*i.e.*, $\sigma_0 + \sigma_b$) is more than the crushing stress σ_c. But in case of long columns, the direct compressive stresses are negligible as compared to buckling stresses. Hence very long columns are subjected to buckling stresses only.

ASSUMPTIONS OF EULER'S COLUMN THEORY

The following assumptions are made in the Euler's column theory :

1. The column is initially perfectly straight and the load is applied axially.
2. The cross-section of the column is uniform throughout its length.
3. The column material is perfectly elastic, homogeneous and isotropic and obeys Hooke's law.
4. The length of the column is very large as compared to its lateral dimensions.
5. The direct stress is very small as compared to the bending stress.
6. The column will fail by buckling alone.
7. The self-weight of column is negligible.

END CONDITION FOR LONG COLUMNS

In case of long columns, the stress due to direct load is very small in comparison with the stress due to buckling. Hence the failure of long columns take place entirely due to buckling (or bending). The following four types of end conditions of the columns are important :

1. Both the ends of the column are hinged (or pinned).
2. One end is fixed and the other end is free.
3. Both the ends of the column are fixed.
4. One end is fixed and the other is pinned.

For a hinged end, the deflection is zero. For a fixed end the deflection and slope are zero. For a free end the deflection is not zero.

19.4.1. Sign Conventions. The following sign conventions for the bending of the columns will be used :

1. A moment which will bend the column with its *convexity* towards its initial central line as shown in Fig. 19.3 (*a*) is taken as positive. In Fig. 19.3 (*a*), *AB* represents the initial centre line of a column. Whether the column bends taking the shape *AB′* or *AB″*, the moment producing this type of curvature is positive.

2. A moment which will tend to bend the column with its *concavity* towards its initial centre line as shown in Fig. 19.3 (*b*) is taken as negative.

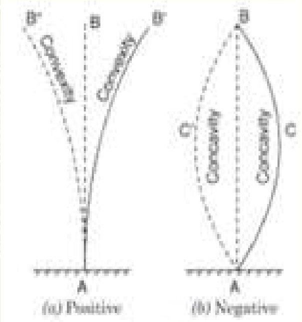

Fig. 19.3

EXPRESSION FOR CRIPPLING LOAD WHEN BOTH THE ENDS OF THE COLUMN ARE HINGED

The load at which the column just buckles (or bends) is called crippling load. Consider a column *AB* of length *l* and uniform cross-sectional area, hinged at both of its ends *A* and *B*. Let *P* be the crippling load at which the column has just buckled. Due to the crippling load, the column will deflect into a curved form *ACB* as shown in Fig. 19.4.

Fig. 19.4

Consider any section at a distance x from the end *A*.

Let y = Deflection (lateral displacement) at the section.

The moment due to the crippling load at the section $= -P \cdot y$

(– ve sign is taken due to sign convention given in Art. 19.4.1)

But moment $= EI \dfrac{d^2y}{dx^2}$.

Equating the two moments, we have

$$EI \frac{d^2y}{dx^2} = -P \cdot y \quad \text{or} \quad EI \frac{d^2y}{dx^2} + P \cdot y = 0$$

or
$$\frac{d^2y}{dx^2} + \frac{P}{EI} \cdot y = 0$$

The solution* of the above differential equation is

$$y = C_1 \cdot \cos\left(x\sqrt{\frac{P}{EI}}\right) + C_2 \cdot \sin\left(x\sqrt{\frac{P}{EI}}\right) \qquad ...(i)$$

where C_1 and C_2 are the constants of integration. The values of C_1 and C_2 are as follows :

(*i*) At *A*, $x = 0$ and $y = 0$ (See Fig. 19.4)

Substituting these values in equation (*i*), we get

$$\begin{aligned} 0 &= C_1 \cdot \cos 0 + C_2 \sin 0 \\ &= C_1 \times 1 + C_2 \times 0 \qquad (\because \cos 0 = 1 \text{ and } \sin 0 = 0) \\ &= C_1 \end{aligned}$$

$$\therefore \quad C_1 = 0. \qquad ...(ii)$$

(*ii*) At *B*, $x = l$ and $y = 0$ (See Fig. 19.4).

Substituting these values in equation (*i*), we get

$$0 = C_1 \cdot \cos\left(l \times \sqrt{\frac{P}{EI}}\right) + C_2 \cdot \sin\left(l \times \sqrt{\frac{P}{EI}}\right)$$

$$= 0 + C_2 \cdot \sin\left(l \times \sqrt{\frac{P}{EI}}\right) \quad [\because C_1 = 0 \text{ from equation } (ii)]$$

$$= C_2 \sin\left(l\sqrt{\frac{P}{EI}}\right) \quad ...(iii)$$

From equation (*iii*), it is clear that either $C_2 = 0$

or $\sin\left(l\sqrt{\frac{P}{EI}}\right) = 0.$

As $C_1 = 0$, then if C_2 is also equal to zero, then from equation (*i*) we will get $y = 0$. This means that the bending of the column will be zero or the column will not bend at all. Which is not true.

$$\therefore \quad \sin\left(l\sqrt{\frac{P}{EI}}\right) = 0$$

$$= \sin 0 \text{ or } \sin \pi \text{ or } \sin 2\pi \text{ or } \sin 3\pi \text{ or } ...$$

or $$l\sqrt{\frac{P}{EI}} = 0 \text{ or } \pi \text{ or } 2\pi \text{ or } 3\pi \text{ or } ...$$

Taking the least practical value,

$$l\sqrt{\frac{P}{EI}} = \pi$$

or $$P = \frac{\pi^2 EI}{l^2}. \quad ...(19.1)$$

EXPRESSION FOR CRIPPLING LOAD WHEN ONE END OF THE COLUMN IS FIXED AND THE OTHER END IS FREE

Consider a column *AB*, of length *l* and uniform cross-sectional area, fixed at the end *A* and free at the end *B*. The free end will sway sideways when load is applied at free end and curvature in the length *l* will be similar to that of upper half of the column whose both ends are hinged. Let *P* is the crippling load at which the column has just buckled. Due to the crippling load *P*, the column will deflect as shown in Fig. 19.5 in which *AB* is the original position of the column and *AB′*, is the deflected position due to crippling load *P*.

Consider any section at a distance *x* from the fixed end *A*.

Let y = Deflection (or lateral displacement) at the section

a = Deflection at the free end *B*.

Then moment at the section due to the crippling load $= P(a - y)$

(+ve sign is taken due to sign convention given in Art. 19.4.1)

But moment is also $= EI \frac{d^2y}{dx^2}$

$\therefore$ Equating the two moments, we get

$$EI\frac{d^2y}{dx^2} = P(a - y) = P \cdot a - P \cdot y$$

or $$EI\frac{d^2y}{dx^2} + P \cdot y = P \cdot a$$

or $$\frac{d^2y}{dx^2} + \frac{P}{EI} \cdot y = \frac{P}{EI} \cdot a. \quad ...(A)$$

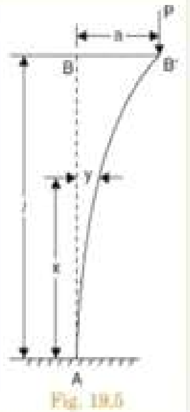

Fig. 19.5

The solution* of the differential equation is

$$y = C_1 \cdot \cos\left(x\sqrt{\frac{P}{EI}}\right) + C_2 \cdot \sin\left(x\sqrt{\frac{P}{EI}}\right) + a \quad ...(i)$$

where C_1 and C_2 are constant of integration. The values of C_1 and C_2 are obtained from boundary conditions. The boundary conditions are :

(*i*) For a fixed end, the deflection as well as slope is zero.

Hence at end A (which is fixed), the deflection $y = 0$ and also slope $\frac{dy}{dx} = 0$.

Hence at A, $x = 0$ and $y = 0$

Substituting these values in equation (*i*), we get

$$0 = C_1 . \cos 0 + C_2 \sin 0 + a$$
$$= C_1 \times 1 + C_2 \times 0 + a \qquad (\because \cos 0 = 1, \sin 0 = 0)$$
$$= C_1 + a$$

$$\therefore \qquad C_1 = -a \qquad ...(ii)$$

At A, $\qquad x = 0$ and $\frac{dy}{dx} = 0$.

Differentiating equation (*i*) w.r.t. x, we get

$$\frac{dy}{dx} = C_1 . (-1) \sin\left(x\sqrt{\frac{P}{EI}}\right) . \sqrt{\frac{P}{EI}} + C_2 \cos\left(x\sqrt{\frac{P}{EI}}\right) . \sqrt{\frac{P}{EI}} + 0$$

$$= -C_1 . \sqrt{\frac{P}{EI}} \sin\left(x\sqrt{\frac{P}{EI}}\right) + C_2 . \sqrt{\frac{P}{EI}} \cos\left(x\sqrt{\frac{P}{EI}}\right)$$

But at A, $x = 0$ and $\frac{dy}{dx} = 0$.

$\therefore$ The above equation becomes as

$$0 = -C_1 . \sqrt{\frac{P}{EI}} \sin 0 + C_2 \sqrt{\frac{P}{EI}} \cos 0$$
$$= -C_1 \sqrt{\frac{P}{EI}} \times 0 + C_2 . \sqrt{\frac{P}{EI}} \times 1 = C_2 \sqrt{\frac{P}{EI}} .$$

From the above equation it is clear that either $C_2 = 0$.

or $\qquad \sqrt{\frac{P}{EI}} = 0.$

But for the crippling load P, the value of $\sqrt{\frac{P}{EI}}$ cannot be equal to zero.

$\therefore \qquad C_2 = 0.$

Substituting the values of $C_1 = -a$ and $C_2 = 0$ in equation (*i*), we get

$$y = -a . \cos\left(x\sqrt{\frac{P}{EI}}\right) + a. \qquad ...(iii)$$

But at the free end of the column, $x = l$ and $y = a$.

Substituting these values in equation (*iii*), we get

$$a = -a . \cos\left(l . \sqrt{\frac{P}{EI}}\right) + a$$

or $$0 = -a . \cos\left(l . \sqrt{\frac{P}{EI}}\right) \text{ or } a \cos\left(l . \sqrt{\frac{P}{EI}}\right) = 0$$

But 'o' cannot be equal to zero

$$\therefore \quad \cos\left(l \cdot \sqrt{\frac{P}{EI}}\right) = 0 = \cos\frac{\pi}{2} \text{ or } \cos\frac{3\pi}{2} \text{ or } \cos\frac{5\pi}{2} \text{ or } \ldots\ldots$$

$$\therefore \quad l \cdot \sqrt{\frac{P}{EI}} = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \text{ or } \frac{5\pi}{2} \ldots\ldots$$

Taking the least practical value,

$$l \cdot \sqrt{\frac{P}{EI}} = \frac{\pi}{2} \quad \text{or} \quad \sqrt{\frac{P}{EI}} = \frac{\pi}{2l}$$

or $$P = \frac{\pi^2 EI}{4l^2} \quad \ldots(19.2)$$

EXPRESSION FOR CRIPPLING LOAD WHEN BOTH THE ENDS OF THE COLUMN ARE FIXED

Consider a column AB of length l and uniform cross-sectional area fixed at both of its ends A and B as shown in Fig. 19.6. Let P is the crippling load at which the column has buckled.

Due to the crippling load P, the column will deflect as shown in Fig. 19.6. Due to fixed ends, there will be fixed end moments (say M_0) at the ends A and B. The fixed end moments will be acting in such direction so that slope at the fixed ends becomes zero.

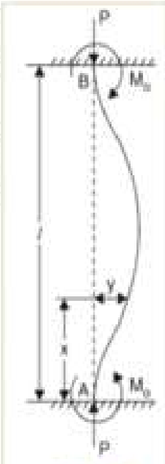

Fig. 19.6

Consider a section at a distance x from the end A. Let the deflection of the column at the section is y. As both the ends of the column are fixed and the column carries a crippling load, there will be some fixed end moments at A and B.

Let M_0 = Fixed end moments at A and B.

Then moment at the section $= M_0 - P.y$

But moment at the section is also $= EI \dfrac{d^2y}{dx^2}$

$\therefore$ Equating the two moments, we get

$$EI \frac{d^2y}{dx^2} = M_0 - P.y$$

or $$EI \frac{d^2y}{dx^2} + P \cdot y = M_0$$

or $$\frac{d^2y}{dx^2} + \frac{P}{EI} \cdot y = \frac{M_0}{EI} \quad \ldots(A)$$

$$= \frac{M_0}{EI} \times \frac{P}{P} = \frac{P}{EI} \cdot \frac{M_0}{P}$$

The solution* of the above differential equation is

$$y = C_1 \cdot \cos\left(x \cdot \sqrt{\frac{P}{EI}}\right) + C_2 \cdot \sin\left(x \cdot \sqrt{\frac{P}{EI}}\right) + \frac{M_0}{P} \quad \ldots(i)$$

where C_1 and C_2 are constant of integration and their values are obtained from boundary conditions. Boundary conditions are :

(i) At A, $x = 0$, $y = 0$ and also $\dfrac{dy}{dx} = 0$ as A is a fixed end.

(ii) At B, $x = l$, $y = 0$ and also $\dfrac{dy}{dx} = 0$ as B is also a fixed end.

Substituting the value $x = 0$ and $y = 0$ in equation (i), we get

$$0 = C_1 \times 1 + C_2 \times 0 + \frac{M_0}{P} \qquad (\because \cos 0 = 1)$$

$$= C_1 + \frac{M_0}{P}$$

$$\therefore \qquad C_1 = -\frac{M_0}{P} \qquad ...(ii)$$

Differentiating equation (i), with respect to x, we get

$$\frac{dy}{dx} = C_1 . (-1) \sin\left(x . \sqrt{\frac{P}{EI}}\right) . \sqrt{\frac{P}{EI}} + C_2 \cos\left(x . \sqrt{\frac{P}{EI}}\right) . \sqrt{\frac{P}{EI}} + 0$$

$$= -C_1 \sin\left(x . \sqrt{\frac{P}{EI}}\right) . \sqrt{\frac{P}{EI}} + C_2 \cos\left(x . \sqrt{\frac{P}{EI}}\right) . \sqrt{\frac{P}{EI}}$$

Substituting the value $x = 0$ and $\frac{dy}{dx} = 0$, the above equation becomes

$$0 = -C_1 \times 0 + C_2 \times 1 \times \sqrt{\frac{P}{EI}} \qquad (\because \sin 0 = 0 \text{ and } \cos 0 = 1)$$

$$= C_2 \sqrt{\frac{P}{EI}} .$$

From the above equation, it is clear that either $C_2 = 0$ or $\sqrt{\frac{P}{EI}} = 0$. But for a given crippling

ad P, the value of $\sqrt{\frac{P}{EI}}$ cannot be equal to zero.

$$\therefore \qquad C_2 = 0.$$

Now substituting the values of $C_1 = -\frac{M_0}{P}$ and $C_2 = 0$ in equation (i), we get

$$y = -\frac{M_0}{P} \cos\left(x \sqrt{\frac{P}{EI}}\right) + 0 + \frac{M_0}{P}$$

$$= -\frac{M_0}{P} \cos\left(x \sqrt{\frac{P}{EI}}\right) + \frac{M_0}{P} \qquad ...(iii)$$

At the end B of the column, $x = l$ and $y = 0$.

Substituting these values in equation (iii), we get

$$0 = -\frac{M_0}{P} \cos\left(l . \sqrt{\frac{P}{EI}}\right) + \frac{M_0}{P}$$

or $$\frac{M_0}{P} \cos\left(l . \sqrt{\frac{P}{EI}}\right) = \frac{M_0}{P}$$

or $$\cos\left(l . \sqrt{\frac{P}{EI}}\right) = \frac{M_0}{P} \times \frac{P}{M_0} = 1 = \cos 0, \cos 2\pi, \cos 4\pi, \cos 6\pi, ...$$

$$\therefore \qquad l \cdot \sqrt{\frac{P}{EI}} = 0, 2\pi, 4\pi, 6\pi, \ldots\ldots$$

Taking the least practical value,

$$l \cdot \sqrt{\frac{P}{EI}} = 2\pi \quad \text{or} \quad P = \frac{4\pi^2 EI}{l^2}. \qquad \ldots(19.3)$$

EXPRESSION FOR CRIPPLING LOAD WHEN ONE END OF THE COLUMN IS FIXED AND THE OTHER END IS HINGED (OR PINNED)

Consider a column AB of length l and uniform cross-sectional area fixed at the end A and hinged at the end B as shown in Fig. 19.7. Let P is the crippling load at which the column has buckled. Due to the crippling load P, the column will deflect as shown in Fig. 19.7.

There will be fixed end moment (M_0) at the fixed end A. This will try to bring back the slope of deflected column zero at A. Hence it will be acting anticlockwise at A. The fixed end moment M_0 at A is to be balanced. This will be balanced by a horizontal reaction (H) at the top end B as shown in Fig. 19.7.

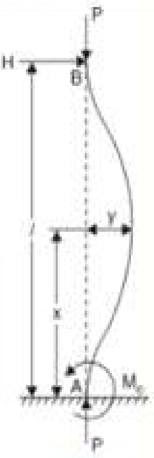

Fig. 19.7

Consider a section at a distance x from the end A.

Let y = Deflection of the column at the section,

M_0 = Fixed end moment at A, and

H = Horizontal reaction at B.

The moment at the section = Moment due to crippling load at B + Moment due to horizontal reaction at B

$$= -P \cdot y + H \cdot (l - x)$$

But the moment at the section is also

$$= EI \frac{d^2y}{dx^2}$$

Equating the two moments, we get

$$EI \frac{d^2y}{dx^2} = -P \cdot y + H(l - x)$$

or

$$EI \frac{d^2y}{dx^2} + P \cdot y = H(l - x)$$

or

$$\frac{d^2y}{dx^2} + \frac{P}{EI} \cdot y = \frac{H}{EI}(l - x) \qquad \text{(Dividing by } EI) \quad \ldots(A)$$

$$= \frac{H}{EI}(l - x) \times \frac{P}{P} = \frac{P}{EI} \cdot \frac{H(l - x)}{P}$$

The solution* of the above differential equation is

$$y = C_1 \cos\left(x\sqrt{\frac{P}{EI}}\right) + C_2 \sin\left(x\sqrt{\frac{P}{EI}}\right) + \frac{H}{P}(l - x) \qquad \ldots(i)$$

where C_1 and C_2 are constants of integration and their values are obtained from boundary conditions. Boundary conditions are :

(i) At the fixed end A, $x = 0$, $y = 0$ and also $\frac{dy}{dx} = 0$

(ii) At the hinged end B, $x = l$ and $y = 0$.

Substituting the value $x = 0$ and $y = 0$ in equation (i), we get

$$0 = C_1 \times 1 + C_2 \times 0 + \frac{H}{P}(l - 0) = C_1 + \frac{H.l}{P}$$

$$\therefore \qquad C_1 = -\frac{H}{P}.l \qquad \text{...(ii)}$$

Differentiating the equation (i) w.r.t. x, we get

$$\frac{dy}{dx} = C_1(-1)\sin\left(x.\sqrt{\frac{P}{EI}}\right).\sqrt{\frac{P}{EI}} + C_2\cos\left(x.\sqrt{\frac{P}{EI}}\right).\sqrt{\frac{P}{EI}} - \frac{H}{P}$$

$$= -C_1\sin\left(x.\sqrt{\frac{P}{EI}}\right).\sqrt{\frac{P}{EI}} + C_2\cos\left(x.\sqrt{\frac{P}{EI}}\right).\sqrt{\frac{P}{EI}} - \frac{H}{P}$$

At A, $x = 0$ and $\frac{dy}{dx} = 0$.

$$\therefore \qquad 0 = -C_1 \times 0 + C_2 .1.\sqrt{\frac{P}{EI}} - \frac{H}{P} \qquad (\because \sin 0 = 0, \cos 0 = 1)$$

$$= C_2\sqrt{\frac{P}{EI}} - \frac{H}{P} \quad \text{or} \quad C_2 = \frac{H}{P}\sqrt{\frac{EI}{P}}.$$

Substituting the values of $C_1 = -\frac{H}{P}.l$ and $C_2 = \frac{H}{P}\sqrt{\frac{EI}{P}}$ in equation (i), we get

$$y = -\frac{H}{P}.l\cos\left(x\sqrt{\frac{P}{EI}}\right) + \frac{H}{P}\sqrt{\frac{EI}{P}}\sin\left(x\sqrt{\frac{P}{EI}}\right) + \frac{H}{P}(l - x)$$

At the end B, $x = l$ and $y = 0$.

Hence the above equation becomes as

$$0 = -\frac{H}{P}l\cos\left(l\sqrt{\frac{P}{EI}}\right) + \frac{H}{P}\sqrt{\frac{EI}{P}}\sin\left(l\sqrt{\frac{P}{EI}}\right) + \frac{H}{P}(l - l)$$

$$= -\frac{H}{P}l\cos\left(l\sqrt{\frac{P}{EI}}\right) + \frac{H}{P}\sqrt{\frac{EI}{P}}\sin\left(l\sqrt{\frac{P}{EI}}\right) + 0$$

or $$\frac{H}{P}\sqrt{\frac{EI}{P}}\sin\left(l\sqrt{\frac{P}{EI}}\right) = \frac{H}{P}l\cos\left(l\sqrt{\frac{P}{EI}}\right)$$

or $$\sin\left(l\sqrt{\frac{P}{EI}}\right) = \frac{H}{P}.l \times \frac{P}{H} \times \sqrt{\frac{P}{EI}}.\cos\left(l\sqrt{\frac{P}{EI}}\right)$$

$$= l.\sqrt{\frac{P}{EI}}.\cos\left(l.\sqrt{\frac{P}{EI}}\right)$$

or $$\tan\left(l\sqrt{\frac{P}{EI}}\right) = l.\sqrt{\frac{P}{EI}}.$$

The solution to the above equation is, $l.\sqrt{\frac{P}{EI}} = 4.5$ radians

Squaring both sides, we get

$$l^2 \cdot \frac{P}{EI} = 4.5^2 = 20.25$$

$$\therefore \qquad P = 20.25\,\frac{EI}{l^2}$$

But approximately $20.25 = 2\pi^2$

$$\therefore \qquad P = \frac{2\pi^2 EI}{l^2} \qquad ...(19.4)$$

19.9. EFFECTIVE LENGTH (OR EQUIVALENT LENGTH) OF A COLUMN

The effective length of a given column with given end conditions is the length of an equivalent column of the same material and cross-section with hinged ends, and having the value of the crippling load equal to that of the given column. Effective length is also called equivalent length.

Let L_e = Effective length of a column,

l = Actual length of the column, and

P = Crippling load for the column.

Then the crippling load for any type of end condition is given by

$$P = \frac{\pi^2 EI}{L_e^{\,2}}. \qquad ...(19.5)$$

The crippling load (P) in terms of actual length and effective length and also the relation between effective length and actual length are given in Table 19.1.

TABLE 19.1

S.No.	End conditions of column	Crippling load in terms of		Relation between effective length and actual length
		Actual length	Effective length	
1.	Both ends hinged	$\frac{\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{L_e^2}$	$L_e = l$
2.	One end is fixed and other is free	$\frac{\pi^2 EI}{4l^2}$	$\frac{\pi^2 EI}{L_e^2}$	$L_e = 2l$
3.	Both ends fixed	$\frac{4\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{L_e^2}$	$L_e = \frac{l}{2}$
4.	One end fixed and other is hinged	$\frac{4\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{L_e^2}$	$L_e = \frac{l}{\sqrt{2}}$

There are two values of moment of inertia *i.e.*, I_{xx} and I_{yy}.

The value of I (moment of inertia) in the above expressions should be taken as the least value of the two moments of inertia as the column will tend to bend in the direction of least moment of inertia.

19.9.1. Crippling Stress in Terms of Effective Length and Radius of Gyration. The moment of inertia (I) can be expressed in terms of radius of gyration (k) as

$$I = Ak^2 \quad \text{where } A = \text{Area of cross-section.}$$

As I is the least value of moment of inertia, then

$$k = \text{Least radius of gyration of the column section.}$$

Now crippling load P in terms of effective length is given by

$$P = \frac{\pi^2 EI}{L_e^{\,2}} = \frac{\pi^2 E \times Ak^2}{L_e^{\,2}} \qquad (\because \; I = Ak^2)$$

$$= \frac{\pi^2 E \times A}{\frac{L_e^{\,2}}{k^2}} = \frac{\pi^2 E \times A}{\left(\frac{L_e}{k}\right)^2} \qquad ...(19.6)$$

And the stress corresponding to crippling load is given by

$$\text{Crippling stress} = \frac{\text{Crippling load}}{\text{Area}} = \frac{P}{A}$$

$$= \frac{\pi^2 E \times A}{A\left(\frac{L_e}{k}\right)^2} \qquad \text{(Substituting the value of } P\text{)}$$

$$= \frac{\pi^2 E}{\left(\frac{L_e}{k}\right)^2} \qquad ...(19.7)$$

19.9.2. Slenderness Ratio. The ratio of the actual length of a column to the least radius of gyration of the column, is known as slenderness ratio.

Mathematically, slenderness ratio is given by

$$\text{Slenderness ratio} = \frac{\text{Actual length}}{\text{Least radius of gyration}} = \frac{l}{k} \qquad ...(19.8)$$

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LECTURER NOTES
3RD SEMESTER
MECHANICAL ENGINEERING

2022

Strength of Material

Submitted by :

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Chapter- 07

TORSION

A shaft is said to be in torsion, when equal and opposite torques are applied at the two ends of the shaft. The torque is equal to the product of the force applied (tangentially to the ends of a shaft) and radius of the shaft. Due to the application of the torques at the two ends, the shaft is subjected to a twisting moment. This causes the shear stresses and shear strains in the material of the shaft.

DERIVATION OF SHEAR STRESS PRODUCED IN A CIRCULAR SHAFT DUE TO TORSION

When a circular shaft is subjected to torsion, shear stresses are set up in the material of the shaft. To determine the magnitude of shear stress at any point on the shaft, consider a shaft fixed at one end *AA* and free at the end *BB* as shown in Fig. 16.1. Let *CD* is any line on the outer surface of the shaft. Now let the shaft is subjected to a torque *T* at the end *BB* as shown in Fig. 16.2. As a result of this torque *T*, the shaft at the end *BB* will rotate clockwise and every cross-section of the shaft will be subjected to shear stresses. The point *D* will shift to *D'* and hence line *CD* will be deflected to *CD'* as shown in Fig. 16.2 (*a*). The line *OD* will be shifted to *OD'* as shown in Fig. 16.2 (*b*).

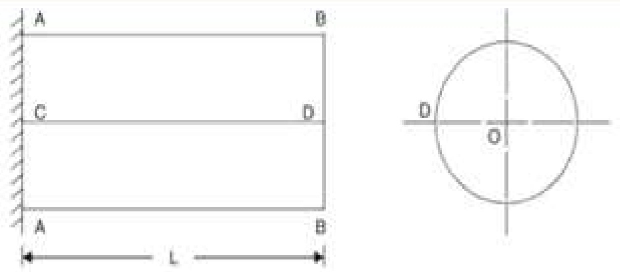

Fig. 16.1. Shaft fixed at one end *AA* before torque *T* is applied.

Let R = Radius of shaft
L = Length of shaft
T = Torque applied at the end BB
τ = Shear stress induced at the surface of the shaft due to torque T
C = Modulus of rigidity of the material of the shaft
ϕ = $\angle DCD'$ also equal to shear strain

$\theta = \angle DOD'$ and is also called angle of twist.

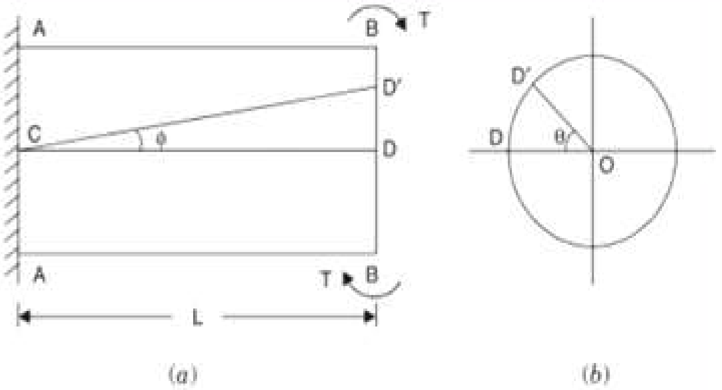

Fig. 16.2. Shaft fixed at AA and subjected to torque T at BB.

Now distortion at the outer surface due to torque T

$$= DD'$$

∴ Shear strain at outer surface

$$= \text{Distortion per unit length}$$

$$= \frac{\text{Distortion at the outer surface}}{\text{Length of shaft}} = \frac{DD'}{L}$$

$$= \frac{DD'}{CD} = \tan\phi$$

$$= \phi \qquad \text{(if } \phi \text{ is very small then } \tan\phi \simeq \phi)$$

∴ Shear strain at outer surface,

$$\phi = \frac{DD'}{L} \qquad ...(i)$$

Now from Fig. 16.2 (*b*).

Arc $DD' = OD \times \theta = R\theta$ $\qquad$ ($\because$ $OD = R$ = Radius of shaft)

Substituting the value of DD' in equation (*i*), we get

Shear strain at outer surface

$$\phi = \frac{R \times \theta}{L} \qquad ...(ii)$$

Now the modulus of rigidity (C) of the material of the shaft is given as

$$C = \frac{\text{Shear stress induced}}{\text{Shear strain produced}} = \frac{\text{Shear stress at the outer surface}}{\text{Shear strain at outer surface}}$$

$$= \frac{\tau}{\left(\frac{R\theta}{L}\right)} \qquad \left(\because \text{ From equation } (ii)\text{, shear strain} = \frac{R\theta}{L}\right)$$

$$= \frac{\tau \times L}{R\theta}$$

$$\therefore \qquad \frac{C\theta}{L} = \frac{\tau}{R} \qquad ...(16.1)$$

$$\therefore \qquad \tau = \frac{R \times C \times \theta}{L}$$

Now for a given shaft subjected to a given torque (*T*), the values of *C*, θ and *L* are constant. Hence shear stress produced is proportional to the radius *R*.

$$\therefore \qquad \tau \propto R \quad \text{or} \quad \frac{\tau}{R} = \text{constant} \qquad ...(iii)$$

If *q* is the shear stress induced at a radius '*r*' from the centre of the shaft then

$$\frac{\tau}{R} = \frac{q}{r} \qquad ...(16.2)$$

But

$$\frac{\tau}{R} = \frac{C\theta}{L} \text{ from equation (16.1)}$$

$$\therefore \qquad \frac{\tau}{R} = \frac{C\theta}{L} = \frac{q}{r}. \qquad ...(16.3)$$

From equation (*iii*), it is clear that shear stress at any point in the shaft is proportional to the distance of the point from the axis of the shaft. Hence the shear stress is maximum at the outer surface and shear stress is zero at the axis of the shaft.

16.2.1. Assumptions Made in the Derivation of Shear Stress Produced in a Circular Shaft Subjected to Torsion. The derivation of shear stress produced in a circular shaft subjected to torsion, is based on the following assumptions :

1. The material of the shaft is uniform throughout.
2. The twist along the shaft is uniform.
3. The shaft is of uniform circular section throughout.
4. Cross-sections of the shaft, which are plane before twist remain plain after twist.
5. All radii which are straight before twist remain straight after twist.

MAXIMUM TORQUE TRANSMITTED BY A SOLID CIRCULAR SHAFT

The maximum torque transmitted by a circular solid shaft, is obtained from the maximum shear stress induced at the outer surface of the solid shaft. Consider a shaft subjected to a torque *T* as shown in Fig. 16.3.

Let τ = Maximum shear stress induced at the outer surface

R = Radius of the shaft

q = Shear stress at a radius '*r*' from the centre.

Consider an elementary circular ring of thickness '*dr*' at a distance '*r*' from the centre as shown in Fig. 16.3. Then the area of the ring,

$$dA = 2\pi r dr$$

From equation (16.2), we have

$$\frac{\tau}{R} = \frac{q}{r}$$

$\therefore$ Shear stress at the radius *r*,

$$q = \frac{\tau}{R} r = \tau \frac{r}{R}$$

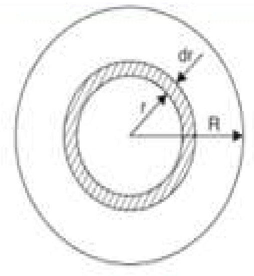

Fig. 16.3

∴ Turning force on the elementary circular ring

$$= \text{Shear stress acting on the ring} \times \text{Area of ring}$$

$$= q \times dA$$

$$= \tau \times \frac{r}{R} \times 2\pi r dr \qquad \left(\because\ q = \tau \times \frac{r}{R}\right)$$

$$= \frac{\tau}{R} \times 2\pi r^2 dr$$

Now turning moment due to the turning force on the elementary ring,

$$dT = \text{Turning force on the ring} \times \text{Distance of the ring from the axis}$$

$$= \frac{\tau}{R} \times 2\pi r^2 dr \times r$$

$$= \frac{\tau}{R} \times 2\pi r^3 dr \qquad ...[16.3\ (A)]$$

∴ The total turning moment (or total torque) is obtained by integrating the above equation between the limits 0 and R

$$\therefore \qquad T = \int_0^R dT = \int_0^R \frac{\tau}{R} \times 2\pi r^3 dr$$

$$= \frac{\tau}{R} \times 2\pi \int_0^R r^3 dr = \frac{\tau}{R} \times 2\pi \left[\frac{r^4}{4}\right]_0^R$$

$$= \frac{\tau}{R} \times 2\pi \times \frac{R^4}{4} = \tau \times \frac{\pi}{2} \times R^3$$

$$= \tau \times \frac{\pi}{2} \times \left(\frac{D}{2}\right)^3 \qquad \left(\because\ R = \frac{D}{2}\right)$$

$$= \tau \times \frac{\pi}{2} \times \frac{D^3}{8} = \tau \times \frac{\pi D^3}{16} = \frac{\pi}{16}\tau D^3 \qquad ...(16.4)$$

Problem 16.1. *A solid shaft of 150 mm diameter is used to transmit torque. Find the maximum torque transmitted by the shaft if the maximum shear stress induced to the shaft is 45 N/mm².*

Sol. Given :

Diameter of the shaft, $D = 150$ mm

Maximum shear stress, $\tau = 45$ N/mm²

Let T = Maximum torque transmitted by the shaft.

Using equation (16.4), $T = \frac{\pi}{16}\tau D^3 = \frac{\pi}{16} \times 45 \times 150^3$

= 29820586 N-mm = **29820.586 N-m. Ans.**

Problem 16.2. *The shearing stress of a solid shaft is not to exceed 40 N/mm² when the torque transmitted is 20000 N-m. Determine the minimum diameter of the shaft.*

Sol. Given :

Maximum shear stress, $\tau = 40$ N/mm²

Torque transmitted, $T = 20000$ N-m $= 20000 \times 10^3$ N-mm

Let D = Minimum diameter of the shaft in mm.

Using equation (16.4),

$$T = \frac{\pi}{16} \tau D^3$$

or
$$D = \left(\frac{16T}{\pi\tau}\right)^{1/3} = \left(\frac{16 \times 20000 \times 10^3}{\pi \times 40}\right)^{1/3} = \mathbf{136.2\ mm.} \quad \text{Ans.}$$

MAXIMUM TORQUE TRANSMITTED BY A HOLLOW CIRCULAR SHAFT

Torque transmitted by a hollow circular shaft is obtained in the same way as for a solid shaft. Consider a hollow shaft. Let it is subjected to a torque T as shown in Fig. 16.4. Take an elementary circular ring of thickness 'dr' at a distance r from the centre as shown in Fig. 16.4.

Let R_0 = Outer radius of the shaft

R_i = Inner radius of the shaft

r = Radius of elementary circular ring

dr = Thickness of the ring

τ = Maximum shear stress induced at outer surface of the shaft

q = Shear stress induced on the elementary ring

dA = Area of the elementary circular ring

$= 2\pi r \times dr$

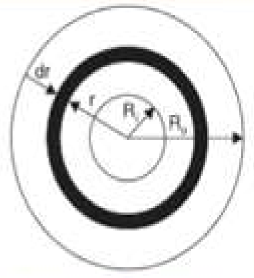

Fig. 16.4. Hollow shaft.

Shear stress at the elementary ring is obtained from equation (16.2) as

$$\frac{\tau}{R_0} = \frac{q}{r} \qquad (\because \text{ Here outer radius } R = R_0)$$

$$\therefore \quad q = \frac{\tau}{R_0} \times r$$

$\therefore$ Turning force on the ring $= \text{Stress} \times \text{Area} = q \times dA$

$$= \frac{\tau}{R_0} r \times 2\pi r dr \qquad \left(\because\ q = \frac{\tau}{R_0} r\right)$$

$$= 2\pi \frac{\tau}{R_0} r^2\, dr$$

Turning moment (dT) on the ring,

dT = Turning force × Distance of the ring from centre

$$= 2\pi \frac{\tau}{R_0} r^2 dr \times r = 2\pi \frac{\tau}{R_0} r^3\, dr$$

The total turning moment (or total torque T) is obtained by integrating the above equation between the limits R_i and R_0.

$$\therefore \quad T = \int_{R_i}^{R_0} dT = \int_{R_i}^{R_0} 2\pi \frac{\tau}{R_0} r^3\, dr$$

$$= 2\pi \frac{\tau}{R_0} \int_{R_i}^{R_0} r^3\, dr$$

($\because$ τ and R_0 are constant and can be taken outside the integral)

$$= 2\pi \frac{\tau}{R_0}\left[\frac{r^4}{4}\right]_{R_i}^{R_0} = 2\pi \frac{\tau}{R_0}\left[\frac{R_0^4 - R_i^4}{4}\right]$$

$$= \frac{\pi}{2}\tau\left[\frac{R_0^4 - R_i^4}{R_0}\right] \quad \text{...(16.5)}$$

Let D_0 = Outer diameter of the shaft

D_i = Inner diameter of the shaft.

Then $R_0 = \frac{D_0}{2}$ and $R_i = \frac{D_i}{2}$.

Substituting the values of R_0 and R_i in equation (16.5),

$$T = \frac{\pi}{2}\tau\left[\frac{\left(\frac{D_0}{2}\right)^4 - \left(\frac{D_i}{2}\right)^4}{\left(\frac{D_0}{2}\right)}\right] = \frac{\pi}{2}\tau\left[\frac{\frac{D_0^4}{16} - \frac{D_i^4}{16}}{\frac{D_0}{2}}\right]$$

$$= \frac{\pi}{2}\tau\left[\frac{D_0^4 - D_i^4}{16} \times \frac{2}{D_0}\right]$$

$$= \frac{\pi}{16}\tau\left[\frac{D_0^4 - D_i^4}{D_0}\right] \quad \text{...(16.6)}$$

POWER TRANSMITTED BY SHAFTS

Once the expression for torque (T) for a solid or a hollow shaft is obtained, power transmitted by the shafts can be determined.

Let N = r.p.m. of the shaft

T = Mean torque transmitted in N-m

ω = Angular speed of shaft.

Then Power = $\frac{2\pi N T^*}{60}$ watts ...(16.7)

$$= \omega \times T \quad \left(\because \frac{2\pi N}{60} = \omega\right)$$

$$= T \times \omega \quad \text{...[16.7 (A)]}$$

Problem 16.3. *In a hollow circular shaft of outer and inner diameters of 20 cm and 10 cm respectively, the shear stress is not to exceed 40 N/mm². Find the maximum torque which the shaft can safely transmit.*

Sol. Given :

Outer diameter, D_0 = 20 cm = 200 mm

Inner diameter, D_i = 10 cm = 100 mm

Maximum shear stress, τ = 40 N/mm^2

Let T = Maximum torque transmitted by the shaft.

Using equation (16.6),

$$T = \frac{\pi}{16}\tau\left[\frac{D_0^4 - D_i^4}{D_0}\right] = \frac{\pi}{16} \times 40\left[\frac{200^4 - 100^4}{200}\right]$$

$$= \frac{\pi}{16} \times 40 \left[\frac{16 \times 10^8 - 1 \times 10^8}{200} \right] = 58904860 \text{ N-mm}$$

$= \mathbf{58904.86}$ **N-m. Ans.**

Problem 16.4. *Two shafts of the same material and of same lengths are subjected to the same torque, if the first shaft is of a solid circular section and the second shaft is of hollow circular section, whose internal diameter is 2/3 of the outside diameter and the maximum shear stress developed in each shaft is the same, compare the weights of the shafts.*

Sol. Given :

Two shafts of the same material and same lengths (one is solid and other is hollow) transmit the same torque and develops the same maximum stress.

Let T = Torque transmitted by each shaft

τ = Max. shear stress developed in each shaft

D = Outer diameter of the solid shaft

D_0 = Outer diameter of the hollow shaft

D_i = Inner diameter of the hollow shaft = $\frac{2}{3} D_0$

W_s = Weight of the solid shaft

W_h = Weight of the hollow shaft

L = Length of each shaft

w = Weight density of the material of each shaft.

Torque transmitted by the solid shaft is given by equation (16.4)

$$T = \frac{\pi}{16} \tau D^3 \qquad ...(i)$$

Torque transmitted by the hollow shaft is given by equation (16.6),

$$T = \frac{\pi}{16} \tau \left[\frac{D_0^4 - D_i^4}{D_0} \right] = \frac{\pi}{16} \tau \left[\frac{D_0^4 - (2/3\, D_0)^4}{D_0} \right]$$

$$= \frac{\pi}{16} \tau \left[\frac{D_0^4 - \frac{16}{81} D_0^4}{D_0} \right] = \frac{\pi}{16} \tau \times \frac{65\, D_0^4}{81 \times D_0}$$

$$= \frac{\pi}{16} \tau \times \frac{65\, D_0^3}{81} \qquad ...(ii)$$

As torque transmitted by solid and hollow shafts are equal, hence equating equations (*i*) and (*ii*),

$$\frac{\pi}{16} \tau D^3 = \frac{\pi}{16} \tau \times \frac{65}{81} D_0^3$$

Cancelling $\frac{\pi}{16} \tau$ from both sides

or
$$D^3 = \frac{65}{81} D_0^3$$

$\therefore$
$$D = \left[\frac{65}{81} D_0^3 \right]^{1/3} = \left(\frac{65}{81} \right)^{1/3} D_0 = 0.929\, D_0 \qquad ...(iii)$$

Now weight of solid shaft, W_s = Weight density × Volume of solid shaft

$= w$ × Area of cross-section × Length

$$= w \times \frac{\pi}{4} D^2 \times L \qquad ...(iv)$$

Weight of hollow shaft,

$$W_h = w \times \text{Area of cross-section of hollow shaft} \times \text{Length}$$

$$= w \times \frac{\pi}{4}[D_0^2 - D_i^2] \times L = w \times \frac{\pi}{4}[D_0^2 - (2/3\,D_0)^2] \times L$$

$$= w \times \frac{\pi}{4}\left[D_0^2 - \frac{4}{9}D_0^2\right] \times L = w \times \frac{\pi}{4} \times \frac{5}{9}D_0^2 \times L \qquad ...(v)$$

Dividing equation (*iv*) by equation (*v*),

$$\frac{W_s}{W_h} = \frac{w \times \frac{\pi}{4}D^2 \times L}{w \times \frac{\pi}{4} \times \frac{5}{9}D_0^2 \times L} = \frac{9D^2}{5D_0^2}$$

$$= \frac{9}{5} \times \frac{(0.929D_0)^2}{D_0^2} \qquad [\because \; D = 0.929\,D_0 \text{ from equation } (iii)]$$

$$= \frac{9}{5} \times 0.929^2 \times \frac{D_0^2}{D_0^2} = \frac{1.55}{1}$$

$$\therefore \quad \frac{\text{Weight of solid shaft}}{\text{Weight of hollow shaft}} = \frac{\mathbf{1.55}}{\mathbf{1}}.$$ **Ans.**

EXPRESSION FOR TORQUE IN TERMS OF POLAR MOMENT OF INERTIA

Polar moment of inertia of a plane area is defined as the moment of inertia of the area about an axis perpendicular to the plane of the figure and passing through the C.G. of the area. It is denoted by symbol J.

The torque in terms of polar moment of inertia (J) is obtained from equation [16.3 (A)] of Art. 16.3.

The moment (dT) on the circular ring is given by equation [16.3 (A)] as

$$dT = \frac{\tau}{R}2\pi r^3 dr = \frac{\tau}{R}2\pi r \times r^2 dr = \frac{\tau}{R}r^2 \times 2\pi r \times dr$$

$$= \frac{\tau}{R}r^2 dA \qquad (\because \; dA = 2\pi r\, dr \text{ see Fig. 16.3})$$

$\therefore$ Total torque, $$T = \int_0^R dT = \int_0^R \frac{\tau}{R}r^2\, dA = \frac{\tau}{R}\int_0^R r^2\, dA \qquad ...(i)$$

But $r^2\,dA$ = Moment of inertia of the elementary ring about an axis perpendicular to the plane of Fig. 16.3 and passing through the centre of the circle.

$\therefore$ $\int_0^R r^2\, dA$ = Moment of inertia of the circle about an axis perpendicular to the plane of the circle and passing through the centre of the circle

$= \text{Polar moment of inertia } (J) = \frac{\pi}{32}D^4.$

Hence equation (*i*) becomes as

$$T = \frac{\tau}{R} \times J \qquad \left(\because\ J = \int_0^R r^2\, dA\right)$$

$$\therefore \qquad \frac{T}{J} = \frac{\tau}{R} \qquad ...(16.8)$$

But from equation (16.1), we have

$$\frac{\tau}{R} = \frac{C\theta}{L}$$

$$\therefore \qquad \frac{T}{J} = \frac{\tau}{R} = \frac{C\theta}{L} \qquad ...(16.9)$$

where C = Modulus of rigidity

θ = Angle of twist in radiation

L = Length of the shaft.

POLAR MODULUS

Polar modulus is defined as the ratio of the polar moment of inertia to the radius of the shaft. It is also called torsional section modulus. It is denoted by Z_p. Mathematically,

$$Z_p = \frac{J}{R}$$

(*a*) For a solid shaft, $J = \frac{\pi}{32} D^4$

$$\therefore \qquad Z_p = \frac{\frac{\pi}{32} D^4}{R} = \frac{\frac{\pi}{32} D^4}{D/2} = \frac{\pi}{16} D^3 \qquad ...(16.10)$$

(*b*) For a hollow shaft, $J = \frac{\pi}{32}(D_0^4 - D_i^4)$ ...(16.11)

$$\therefore \qquad Z_p = \frac{\frac{\pi}{32}[D_0^4 - D_i^4]}{R}$$

(Here R is the outer radius) $\left(\because\ R = \frac{D_0}{2}\right)$

$$= \frac{\frac{\pi}{32}[D_0^4 - D_i^4]}{D_0/2} = \frac{\pi}{16 D_0} \times [D_0^4 - D_i^4] \qquad ...(16.12)$$

STRENGTH OF SHAFT AND TORSIONAL RIGIDITY

The strength of a shaft means the *maximum torque* or *maximum power* the shaft can transmit.

Torsional rigidity or stiffness of the shaft is defined as the product of modulus of rigidity (C) and polar moment of inertia of the shaft (J). Hence mathematically, the torsional rigidity is given as,

Torsional rigidity = $C \times J$.

Torsional rigidity is also defined as the torque required to produce a twist of one radian per unit length of the shaft.

Let a twisting moment T produces a twist of θ radians in a shaft of length L.

Using equation (16.9), we have

$$\frac{T}{J} = \frac{C\theta}{L} \quad \text{or} \quad C \times J = \frac{T \times L}{\theta}$$

But $C \times J$ = Torsional rigidity

$\therefore$ Torsional rigidity $= \frac{T \times L}{\theta}$

If L = one metre and θ = one radian

Then torsional rigidity = Torque.

Problem 16.13. *Determine the diameter of a solid steel shaft which will transmit 90 kW at 160 r.p.m. Also determine the length of the shaft if the twist must not exceed 1° over the entire length. The maximum shear stress is limited to 60 N/mm². Take the value of modulus of rigidity = 8 × 10⁴ N/mm².*

Sol. Given :

Power, $P = 90 \text{ kW} = 90 \times 10^3 \text{ W}$

Speed, $N = 160$ r.p.m.

Angle of twist, $\theta = 1°$ or $\frac{\theta}{180}$ radian $\left(\because 1° = \frac{\pi}{180} \text{ radian}\right)$

Max. shear stress, $\tau = 60 \text{ N/mm}^2$

Modulus of rigidity, $C = 8 \times 10^4 \text{ N/mm}^2$

Let D = Diameter of the shaft and

L = Length of the shaft.

(i) Diameter of the shaft

Using equation (16.7),

$$P = \frac{2\pi NT}{60}$$

or

$$90 \times 10^3 = \frac{2\pi \times 160 \times T}{60}$$

$$\therefore \quad T = \frac{90 \times 10^3 \times 60}{2\pi \times 160} = 5371.48 \text{ N-m} = 5371.48 \times 10^3 \text{ N-mm}$$

Now using equation (16.4),

$$T = \frac{\pi}{16} \tau D^3$$

or

$$5371.48 \times 10^3 = \frac{\pi}{16} \times 60 \times D^3$$

$$\therefore \quad D^3 = \frac{5371.48 \times 10^3 \times 16}{\pi \times 60} = 455945$$

$\therefore$ $D = (455945)^{1/3} = \mathbf{76.8 \text{ mm.}}$ **Ans.**

(ii) Length of the shaft

Using equation (16.7),

$$\frac{\tau}{R} = \frac{C\theta}{L}$$

$$\frac{60}{\left(\frac{76.8}{2}\right)} = \frac{8 \times 10^4 \times \pi}{L \times 180} \quad \left(\because R = \frac{D}{2} = \frac{76.8}{2} \text{ mm}, \theta = \frac{\pi}{180} \text{ radian}\right)$$

$$L = \frac{8 \times 10^4 \times \pi \times 76.8}{60 \times 180 \times 2} = \mathbf{893.6 \text{ mm.}}$$ **Ans.**